

# AGLER-COMMUTANT LIFTING ON AN ANNULUS

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**ABSTRACT.** This note presents a commutant lifting theorem (CLT) of Agler type for the annulus  $\mathbb{A}$ . Here the relevant set of test functions are the minimal inner functions on  $\mathbb{A}$  - those analytic functions on  $\mathbb{A}$  which are unimodular on the boundary and have exactly two zeros in  $\mathbb{A}$  - and the model space is determined by a distinguished member of the Sarason family of kernels over  $\mathbb{A}$ . The ideas and constructions borrow freely from the CLT of Ball, Li, Timotin, and Trent [14] and Archer [11] for the polydisc, and Ambrozie and Eschmeier for the ball in  $\mathbb{C}^n$  [3], as well as generalizations of the de Branges-Rovnyak construction like found in Agler [5] and Ambrozie, Englis, and Müller [4]. It offers a template for extending the result in [29] to infinitely many test functions. Among the needed new ingredients is the formulation of the factorization implicit in the statement of the results in [14], [11] and [29] in terms of certain functional Hilbert spaces of Hilbert space valued functions.

## 1. INTRODUCTION

Results going back to [5] and including [7], [15], [16], [10] [4], [3] [20], [19] among others view the starting point for Agler-Pick interpolation as a collection of functions  $\Psi$ , called test functions. Roughly speaking one constructs an operator algebra whose norm is as large as possible subject to the condition that each  $\psi \in \Psi$  is contractive. The corresponding Agler-Schur class, or  $\Psi$ -Agler-Schur class, is then the unit ball of this operator algebra and interpolation is within this class.

The by now classical example is that of Agler-Pick interpolation in the  $d$ -fold polydisc  $\mathbb{D}^d \subset \mathbb{C}^d$  with  $\Psi = \{z_1, \dots, z_d\}$ , where the  $z_j$  are the coordinate functions [5][7]. In this case the unit ball of the resultant operator algebra of functions on  $\mathbb{D}^d$  is known as the Agler-Schur class, often denoted  $\mathcal{S}_d$ . For  $d = 1, 2$  this operator algebra is the same as  $H^\infty(\mathbb{D}^d)$ , but generally  $\mathcal{S}_d$  and  $H^\infty(\mathbb{D}^d)$  are different. The literature contains many articles on the Agler-Schur class and its operator-valued generalizations. A sample of references include [12] [16][25][8]. Of special relevance for this paper is the work of Ambrozie [10] and the subsequent articles [20] and [19], where the set of test functions  $\Psi$  is allowed to be infinite with a compact Hausdorff topology.

It has long been known that Pick interpolation is a special case of commutant lifting [31] [21] [23] [30]. In this spirit Ball, Li, Timotin, and Trent [14] formulate and prove an Agler-Pick type commutant lifting theorem for the polydisc. Significant refinements of both the statements and proofs of this result appear in the work of Archer [11]. Ambrozie and Eschmeier [3] establish a related CLT for the unit ball in  $\mathbb{C}^n$ . In [29] we establish a generalization of these results to the case of a

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finite collection  $\Psi$  together with a distinguished reproducing kernel Hilbert space  $H^2(k)$ , unlocking the prior tight connection between the coordinate (test) functions  $\{z_1, \dots, z_d\}$  and the kernel  $k$  for the Hardy space  $H^2(\mathbb{D}^d)$  in the case of the polydisc. In this more general context, the lack of an orthonormal basis explicitly expressible in terms of the test functions necessitated a number of innovations.

In this article we pursue an Agler-Pick type commutant lifting theorem with  $\Psi$  the infinite collection of minimal inner functions on an annulus  $\mathbb{A}$  - those with unimodular boundary values and exactly two zeros inside - and  $H^2(k)$  a distinguished choice of Hardy Hilbert space on  $\mathbb{A}$  - distinguished by the fact that  $k(z, w)$  is the only Sarason kernel for  $\mathbb{A}$  which does not vanish for  $(z, w) \in \mathbb{A} \times \mathbb{A}$ . In addition to certain measure theoretic considerations necessitated by the infinite collection of test functions, it also turns out that some structures not apparent or exploited in the case of finite test functions become important. We have borrowed freely from [14], [11], [3], [5] [4] and of course [29].

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## 2. PRELIMINARIES AND MAIN RESULT

Fix  $0 < q < 1$  and let  $\mathbb{A}$  denote the annulus  $\{z \in \mathbb{C} : q < |z| < 1\}$ . The boundary of the annulus comes in two parts, the outer boundary  $B_0 = \{|z| = 1\}$  and the inner boundary  $B_1 = \{|z| = q\}$ . As is customary,  $\mathbb{D}$  denotes the unit disc.

**2.1. The test functions.** The minimal inner functions on  $\mathbb{A}$  are those (non-constant) analytic functions  $\phi : \mathbb{A} \rightarrow \mathbb{D}$  whose boundary values are unimodular and have the minimum number of zeros - two - in  $\mathbb{A}$ . Up to canonical normalizations, they can be parametrized by the unit circle.

If  $\psi : \mathbb{A} \rightarrow \mathbb{D}$  is a minimal inner function normalized by  $\psi(\sqrt{q}) = 0$  and  $\psi(1) = 1$ , then the second zero  $w$  of  $\psi$  must lie on the circle  $\mathbb{T} = \{z : |z| = \sqrt{q}\}$  (see Section 11). Conversely, if  $w$  is a point on this circle  $\mathbb{T}$ , then there is a (uniquely determined) minimal inner function  $\psi_w$  with  $\psi_w(\sqrt{q}) = 0 = \psi_w(w)$  normalized by  $\psi_w(1) = 1$ . In the case  $w = \sqrt{q}$ , this zero has multiplicity two. Hence, letting  $\Psi = \{\psi_w : w \in \mathbb{T}\} \subset H^\infty(\mathbb{A})$ , there is a canonical bijection  $\mathbb{T} \rightarrow \Psi$  given by  $w \mapsto \psi_w$  which turns out to be a homeomorphism.

For  $z \in \mathbb{A}$ , let  $E(z)$  denote the corresponding point evaluation on  $\Psi$ . Thus  $E(z) : \Psi \rightarrow \mathbb{D}$  is the continuous function defined by  $E(z)(\psi) = \psi(z)$ .

**2.2. Transfer functions and the Schur class.** In the test function approach to interpolation and commutant lifting, those functions built from the test functions as a transfer function of a unitary colligation play a key role and are known as Agler-Schur class functions.

**Definition 2.1.** A  $\Psi$ -unitary colligation is a tuple  $\Sigma = (\rho, A, B, C, D, \mathcal{E}, \mathcal{H})$  where

- (i)  $\mathcal{E}$  and  $\mathcal{H}$  are Hilbert spaces;
- (ii)  $\rho : C(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{E})$  is a unital representation; and
- (iii) the block operator

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathcal{E} \\ \oplus \\ \mathcal{H} \end{matrix} \rightarrow \begin{matrix} \mathcal{E} \\ \oplus \\ \mathcal{H} \end{matrix}$$

is unitary.

The corresponding *transfer function* is the function on  $\mathbb{A}$  with values in  $\mathcal{B}(\mathcal{H})$  given by

$$W_\Sigma = D + C(I - ZA)^{-1}ZB,$$

where  $Z : \mathbb{A} \rightarrow \mathcal{B}(\mathcal{E})$  is the function  $\rho(E(z))$ .

The collection  $\mathcal{S}(\mathbb{A}, \mathcal{H})$  of functions  $F : \mathbb{A} \rightarrow \mathcal{B}(\mathcal{H})$  with a transfer function representation is called the Schur-Agler class. It coincides with the usual unit ball of  $H^\infty(\mathbb{A})$  for scalar-valued functions [19]  $\mathcal{H} = \mathbb{C}$ ). We believe that, using Agler's rational dilation theorem [5] and arguments like those in [19] or those of [18], the same is true for operator-valued  $H^\infty(\mathbb{A})$ , but postpone further consideration of this issue.

**2.3. A Hardy space of the annulus.** Results of Sarason [31], Abrahamse and Douglas [2], and Abrahamse [1] among others identify a certain one parameter family of Hardy Hilbert spaces over the annulus which, collectively, play the same role for  $\mathbb{A}$  as the classical Hardy space plays for  $\mathbb{D}$ .

For  $t > 0$ , let  $\mu_t$  denote the measure on the boundary of  $\mathbb{A}$  which is the usual normalized arclength measure on the outer boundary  $B_0$  (so that  $\mu_t(B_0) = 1$ ), but is  $t$  times normalized arclength measure on the inner boundary  $B_1$  (so that  $\mu_t(B_1) = t$ ). Let  $H_t^2 = H_t^2(\mathbb{A})$  denote the Hardy Hilbert space obtained by closing up functions analytic in a neighborhood of the closure of  $\mathbb{A}$  in  $L^2(\mu_t)$ .

It is straightforward to check that the set

$$(1) \quad \zeta_n = \frac{z^n}{\sqrt{1 + tq^{2n}}}, \quad n \in \mathbb{Z},$$

is an orthonormal basis for  $H_t^2$ . In particular,

$$(2) \quad k(z, w; t) = \sum_{n \in \mathbb{Z}} \frac{(zw^*)^n}{1 + tq^{2n}}$$

is the reproducing kernel for  $H_t^2$ .

Each  $\varphi \in H^\infty(\mathbb{A})$  determines an operator  $M_t(\varphi)$  of multiplication by  $\varphi$  on  $H_t^2$  whose adjoint satisfies

$$M_t(\varphi)^* k(\cdot, w; t) = \varphi(w)^* k(\cdot, w; t).$$

From equation (2), it is evident that  $U : H_{q^2t}^2 \mapsto H_t^2$  given by  $Uf = zf$  is unitary. It also intertwines  $M_{tq^2}$  and  $M_t$ ; i.e.,  $UM_{tq^2}(\varphi) = M_t(\varphi)U$ . Modulo this equivalence, the collection  $(H_t^2, M_t)$  is a family of representations of  $H^\infty(\mathbb{A})$  parametrized by the unit circle. Up to unitary equivalence, these are Sarason's Hardy spaces of the annulus [31] that appear in [1]. They are also, over  $\mathbb{A}$ , the rank one bundle shifts of Abrahamse and Douglas [2].

The kernel functions  $k(z, w; t)$  have theta function representations from which the proposition below follows. From here on, let  $k(z, w) = k(z, w; 1)$  and  $H^2(\mathbb{A}) = H_1^2(\mathbb{A})$ . This is our distinguished Hardy space and its kernel. Set  $k_w(z) = k(z, w)$ .

**Proposition 2.2.** *The kernel  $k(\cdot, \cdot)$  doesn't vanish in the annulus; i.e., for  $z, w \in \mathbb{A}$ ,  $k(z, w) \neq 0$ , but it does vanish on the boundary as  $k(1, -1) = 0$ . Further, there is a constant  $C' > 0$  independent of  $z$  and  $w$  in  $\mathbb{A}$  so that*

$$\frac{1}{k(z, w)} = C' k(z, -w).$$

*If  $t \neq q^{2m}$  (for any  $m$ ), then there exists  $z, w \in \mathbb{A}$  such that  $k(z, w; t) = 0$ .*

A proof of the proposition appears in Section 10.

In the sequel, frequent use will be made of the Hilbert space tensor product  $H^2(k) \otimes \mathcal{H}$ , where  $\mathcal{H}$  is itself a Hilbert space. A convenient way to define this Hilbert space is as those (Laurent) series

$$h = \sum_{j \in \mathbb{Z}} \zeta_j \otimes h_j,$$

for which  $\sum \|h_j\|^2$  converges. The inner product is defined by

$$\langle h, g \rangle = \sum \langle h_j, g_j \rangle.$$

For  $z \in \mathbb{A}$ , the sum

$$h(z) = \sum \zeta_j(z) \otimes h_j$$

converges absolutely. It follows that, for a fixed  $g \in \mathcal{H}$ ,

$$\langle h(z), g \rangle_{\mathcal{H}} = \langle h, k_z \otimes g \rangle.$$

A function  $W : \mathbb{A} \rightarrow \mathcal{B}(\mathcal{H})$  defines a contraction operator  $M_W$  on  $H^2(k) \otimes \mathcal{H}$  by

$$(3) \quad M_W^*[k_z \otimes g] = k_z \otimes W(z)^*g$$

if and only if the (operator-valued) kernel

$$\mathbb{A} \times \mathbb{A} \ni (z, w) \mapsto (I - W(z)W(w)^*)k(z, w)$$

is positive semi-definite [9][13]. Because, for  $h \in H^2(k) \otimes \mathcal{H}$ ,

$$\langle M_W h, k_z \otimes g \rangle = \langle W(z)h(z), g \rangle,$$

it is natural to write  $(M_W h)(z) = W(z)h(z) = (Wh)(z)$  to denote the operator  $M_W$  and identify it with the function  $W(z)$ .

The following standard lemma will be used often and without comment in the sequel.

**Lemma 2.3.** *If  $W_n : \mathbb{A} \rightarrow \mathcal{B}(\mathcal{H})$  is a sequence of functions which converge pointwise (in the norm topology) to  $W$  and if  $\{W_n\}$  is uniformly bounded, then  $M_W$  is bounded and the sequence  $(M_{W_n})$  converges WOT to  $W$ .*

For expository purposes, we record the following nice relation between the kernel  $k$  and the test functions.

**Proposition 2.4** ([26, 27]). *For the test function  $\psi$  with zeros  $\sqrt{q}$  and  $w$  with  $|w| = \sqrt{q}$  the kernel*

$$\mathbb{A} \times \mathbb{A} \ni (z, w) \mapsto k(z, w)(1 - \psi(z)\psi(w)^*) = \langle (I - M_\psi M_\psi^*)k(\cdot, w), k(\cdot, z) \rangle$$

*has rank two and is positive semi-definite.*

*Further,  $M_\psi$  is a shift of multiplicity two and the kernel of  $I - M_\psi M_\psi^*$  is the span of  $k(\cdot, \sqrt{q})$  and  $k(\cdot, w)$  (except of course when  $w = \sqrt{q}$  when we must resort to using a derivative).*

**2.4. Some representations and the functional calculus.** Let  $T$  denote an operator on a Hilbert space  $\mathcal{M}$  with  $\sigma(T) \subset \mathbb{A}$ . This spectral condition (as opposed to the more liberal  $\sigma(T) \subset \overline{\mathbb{A}}$ ) is imposed because we wish to consider  $\frac{1}{k}(T, T^*)$  and  $\frac{1}{k}$  does not extend to be analytic in  $z$  and  $w^*$  beyond  $\mathbb{A} \times \mathbb{A}$ . Let  $T$  also denote the corresponding representation  $T : H^\infty(\mathbb{A}) \rightarrow \mathcal{B}(\mathcal{M})$ , given by  $T(f) = f(T)$ . We also use the notation  $T_f = f(T)$ . Note that  $T$  is weakly continuous in the sense that if  $f, f_n \in H^\infty(\mathbb{A})$  and  $f_n$  converges to  $f$  uniformly on compact sets, then  $T_{f_n}$  converges in operator norm to  $T_f$ .

**2.4.1. The hereditary functional calculus.** Given an operator  $T$  and a polynomial  $p(z, w) = \sum p_{j,\ell} z^j (w^*)^\ell$ , the hereditary calculus of Agler [5] evaluates  $p(T, T^*) = \sum p_{j,\ell} T^j (T^*)^\ell$ . The calculus extend to functions  $f(z, w)$  which are analytic in  $z$  and coanalytic in  $w$  on a neighborhood of  $\sigma(T) \times \sigma(T)^*$ . Here we will not need the full power of the calculus, but we do need a generalization like that found in [4]. For integers  $j$ , let  $T_j$  denote  $T_{\zeta_j}$ , where  $\zeta_j$  is defined in equation (1) (with  $t = 1$ ).

For an operator  $T \in \mathcal{B}(\mathcal{M})$  with  $\sigma(T) \subset \mathbb{A}$ , and  $G \in \mathcal{B}(\mathcal{M})$ , the sum

$$k(T, T^*)(G) := \sum_{-\infty}^{\infty} T_j G T_j^*$$

converges absolutely. The same is also true of

$$\frac{1}{k}(T, T^*)(G) := C' \sum_{-\infty}^{\infty} (-1)^j T_j G T_j^*.$$

The following Lemma follows from the functional calculus considerations in [4] together with the fact that, by hypothesis,  $\sigma(T) \times \sigma(T^*) \subset \mathbb{A} \times \mathbb{A}$  (see [22]).

**Lemma 2.5.** *Let  $T, G \in \mathcal{B}(\mathcal{M})$  be given. If  $\sigma(T) \subset \mathbb{A}$ , then*

$$k(T, T^*)\left(\frac{1}{k}(T, T^*)(G)\right) = G,$$

and likewise,

$$\frac{1}{k}(T, T^*)(k(T, T^*)(G)) = G.$$

If  $G_\alpha \in \mathcal{B}(\mathcal{M})$  is a (norm bounded) net which converges WOT to  $G \in \mathcal{B}(\mathcal{M})$ , then  $k(T, T^*)(G_\alpha)$  converges WOT to  $k(T, T^*)(G)$ ; and likewise  $\frac{1}{k}(T, T^*)(G_\alpha)$  converges WOT to  $\frac{1}{k}(T, T^*)(G)$ .

**2.5. The model operator.** The operator of multiplication by  $z$  on  $H^2(k)$  gives rise to the representation  $M : H^\infty(\mathbb{A}) \rightarrow \mathcal{B}(H^2(k))$  defined by  $M(f)g = M_f g = fg$ . (Note  $\sigma(M_\zeta) = \overline{\mathbb{A}}$ .) To simplify notation, if  $\mathcal{H}$  is a Hilbert space, we also use  $M$  to denote the representation  $M \otimes I_{\mathcal{H}}$  on  $H^2(k) \otimes \mathcal{H}$ .

We say that  $M$  on  $H^2(k) \otimes \mathcal{H}$  lifts the representation  $T : H^\infty(\mathbb{A}) \rightarrow \mathcal{B}(\mathcal{M})$  if there is an isometry  $V : \mathcal{M} \rightarrow H^2(k) \otimes \mathcal{H}$  so that  $VT^* = M^*V$ ; i.e., for each  $f \in H^\infty$ ,  $VT_f^* = M_f^*V$ . An application of Runge's Theorem, or simply arguing with Laurent series, together with the considerations in Subsection 2.4 shows that it suffices to assume that  $VT_\zeta^* = M_\zeta^*V$ .

If  $\mathcal{M} \subset H^2(k) \otimes \mathcal{H}$  is invariant for  $M^*$  (that is  $M_f^* \mathcal{M} \subset \mathcal{M}$  for all  $f \in H^\infty(\mathbb{A})$ ), then  $T = V^* M V$  given by  $T_f = P M_f P$ , where  $V$  is the inclusion of  $\mathcal{M}$  into  $H^2(k) \otimes \mathcal{H}$ , is also a representation. Indeed, in this case  $M$  lifts  $T$ .

**2.6. Agler decompositions.** Suppose  $T \in \mathcal{B}(\mathcal{M})$  is an operator with  $\sigma(T) \subset \mathbb{A}$  and such that  $T$  is lifted by  $M$ . Further suppose  $X \in \mathcal{B}(\mathcal{M})$  commutes with  $T$ ; i.e.,  $T_f X = X T_f$  for all  $f \in H^\infty(\mathbb{A})$ . As in Subsection 2.5, note that it suffices to assume that  $T_\zeta X = X T_\zeta$ .

An *Agler decomposition*, for the pair  $(T, X)$  is a  $\mathcal{B}(\mathcal{M})$ -valued measure  $\mu$  on  $\mathfrak{B}(\mathbb{T})$ , the Borel subsets of  $\mathbb{T}$  (identifying  $\Psi$  with  $\mathbb{T}$ ),  $\mu : \mathfrak{B}(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{M})$  such that

(i) for each  $\varphi$  in the scalar Schur class and each Borel set  $\omega$ ,

$$(4) \quad k(T, T^*)(\mu(\omega)) - T_\varphi k(T, T^*)(\mu(\omega)) T_\varphi^* \succeq 0 \text{ and};$$

(ii)

$$(5) \quad \frac{1}{k}(T, T^*)(I - X X^*) = \mu(\mathbb{T}) - \int T_\psi d\mu(\psi) T_\psi^*.$$

Here, for self-adjoint operators  $A$  and  $B$ , the notation  $A \succeq B$  means  $A - B$  is positive semi-definite and similarly  $A \succ B$  means  $A - B$  is positive definite.

Several remarks are in order.

**Remark 2.6.** The integral on the right hand side of item (ii) is interpreted weakly as follows. Given a measurable partition  $P = (\omega_j)_{j=1}^n$  of  $\mathbb{T}$  and points  $S = (s_j \in \omega_j)$ , let  $\Delta(P, S, \mu) = \sum T_{s_j} \mu(\omega_j) T_{s_j}^*$ . The tagged partitions  $(P, S)$  form an directed set ordered by refinement of partitions, and it turns out, because of (4), that the net  $\{\Delta(P, S, \mu) : (P, S)\}$  converges in the WOT and its limit is the integral.

Thus the integral here, and the corresponding  $L^2$  spaces that appear later, shares much with the integration theory based of Riemann sums and is not so different than others found in the literature. For a recent example, see [24]. Detail of the construction are given in Section 4. Narrowly tailoring the development to the present needs has the virtue of keeping the presentation self contained and ultimately the paper shorter.

**Remark 2.7.** The definition of operator-valued measure requires  $\mu$  to be WOT countably additive. Thus, the second part of Lemma 2.5 implies that  $\Lambda(\omega) = k(T, T^*)(\mu(\omega))$  is also an operator-valued measure.

It is not assumed that  $\mu(\mathbb{T}) = I$ .

## 2.7. The main result.

**Definition 2.8.** Given  $T \in \mathcal{B}(\mathcal{M})$  with  $\sigma(T) \subset \mathbb{A}$ , a lifting  $VT^* = M^*V$  of  $T$  by  $M$  on  $H^2(k) \otimes \mathcal{H}$  is *minimal* if  $Q^*V\mathcal{M}$  is dense in  $\mathcal{H}$ . Here  $Q^* \sum f_j \zeta_j = f_0$ .

In the next section it is shown that a minimal lifting is essentially unique. The following theorem is the main result of this paper.

**Theorem 2.9.** *Let  $\mathcal{M}$  be a separable Hilbert space. Suppose  $X, T \in \mathcal{B}(\mathcal{M})$  and*

- (i)  $\sigma(T) \subset \mathbb{A}$ ;
- (ii)  $M$  on  $H^2(k) \otimes \mathcal{H}$  with  $VT^* = M^*V$  is a minimal lifting; and
- (iii)  $XT_\varphi = T_\varphi X$  for each  $\varphi \in H^\infty(\mathbb{A})$ .

*The following are equivalent.*

- (sc) *There is an  $F \in \mathcal{S}(\mathbb{A}, \mathcal{H})$  so that  $XV^* = V^*M_F$ .*
- (ad) *There is an Agler decomposition  $\mu : \mathfrak{B}(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{M})$  for the pair  $(T, X)$ .*

**Remark 2.10.** It is illuminating to consider the special case of Agler-Pick interpolation on  $\mathbb{A}$ . Let  $z_1, \dots, z_n \in \mathbb{A}$  and  $w_1, \dots, w_n \in \mathbb{D}$  be given. Let  $\mathcal{M} \subset H^2(k)$  denote the span of  $\{k_{z_j}\}$  and let  $V$  denote the inclusion of  $\mathcal{M}$  into  $H^2(k)$ . Then  $T$  defined by  $T = V^*MV$  is lifted by  $M$  and its spectrum is the set of  $\{z_j\}$ . Define  $X^*$  on  $\mathcal{M}$  by  $X^*k_{z_j} = w_j^*k_{z_j}$ . Then  $X$  commutes with  $T$ . In this case

$$\left\langle \frac{1}{k}(T, T^*)(I - XX^*)k_{z_\ell}, k_{z_j} \right\rangle = 1 - w_j w_\ell^*$$

and

$$\int T_\psi \langle d\mu(\psi) T_\psi^* k_{z_\ell}, k_{z_j} \rangle = \int \psi(z_j) \psi(z_\ell)^* \langle d\mu(\psi) k_{z_\ell}, k_{z_w} \rangle.$$

Thus part (ii) in an Agler decomposition takes the form,

$$1 - w_j w_\ell^* = \int [1 - \psi(z_j) \psi(z_\ell)^*] \langle d\mu(\psi) k_{z_\ell}, k_{z_w} \rangle.$$

### 3. MORE ON LIFTINGS

Recall the orthonormal basis  $\{\zeta_n\}_{n \in \mathbb{Z}}$  (with  $t = 1$ ) of equation (1) and let  $T_j$  and  $M_j$  denote  $T_{\zeta_j}$  and  $M_{\zeta_j}$  respectively, where  $M^*$  acting on  $H^2(k) \otimes \mathcal{H}$  lifts  $T^*$  acting on  $\mathcal{M}$ .

The following is a version of a theorem of Ambrozie, Englis, and Müller [4], a result very much in the spirit of the de Branges-Rovnyak construction [17] and related to the results of [6].

**Proposition 3.1.** *Suppose  $T \in \mathcal{B}(\mathcal{M})$  and  $\sigma(T) \subset \mathbb{A}$ .*

*If  $M = M \otimes I_{\mathcal{H}}$  lifts  $T$  with  $VT^* = M^*V$ , then*

$$(6) \quad Vh = \sum \zeta_j \otimes RT_j^* h$$

where  $R = Q^*V : \mathcal{M} \rightarrow \mathcal{H}$ , the operator  $Q : \mathcal{H} \rightarrow H^2(k) \otimes \mathcal{H}$  is defined by

$$Q^* \sum f_j \otimes \zeta_j = f_0,$$

and the sum converges in norm. In particular, the (non-decreasing) sum

$$(7) \quad \sum_{j=-n}^n T_j R^* R T_j^*$$

converges WOT to the identity.

Conversely, if there is an  $R : \mathcal{M} \rightarrow \mathcal{H}$  so that the sum in equation (7) converges WOT to the identity, then  $M$  lifts  $T$  via  $VT^* = M^*V$  where  $V$  is given by equation (6).

Moreover, for  $f \in H^\infty$  and  $h \in \mathcal{H}$ ,

$$(8) \quad V^*(f \otimes h) = T_f R^* h.$$

*Proof.* Suppose  $V : \mathcal{M} \rightarrow H^2(k) \otimes \mathcal{H}$  is an isometry and  $VT_f^* = M_f^*V$  for all  $f \in H^\infty(\mathbb{A})$ . Since  $V : \mathcal{M} \rightarrow H^2(k) \otimes \mathcal{H}$ , there exists operators  $R_j : \mathcal{M} \rightarrow \mathcal{H}$  so that, for  $h \in \mathcal{M}$ ,

$$Vh = \sum \zeta_j \otimes R_j h,$$

with the sum converging SOT. Now,

$$\begin{aligned} \sum \zeta_j \otimes R_j T_m^* h &= V T_m^* h \\ &= M_m^* V h \\ &= \sum M_m^* \zeta_j \otimes R_j h. \end{aligned}$$

Taking the inner product of both sides of the above equation with  $1 \otimes e$  ( $e \in \mathcal{H}$ ) gives,

$$\langle R_0 T_m^* h, e \rangle = \langle R_m h, e \rangle.$$

With  $R = R_0$ , this shows  $R_m = R T_m^*$  and thus proves that  $V$  takes the form promised in equation (6). That this sum converges in norm follows from the spectral condition on  $T$ .

To prove the conversely, the hypothesis that the sum converges WOT to the identity implies that  $V$  defined as in equation (6) (which converges in norm) is an isometry. We next prove equation (8), from which the conclusion that  $M$  lifts  $T$  via  $V T^* = M^* V$  will follow.

To start, note that, for each  $m \in \mathbb{Z}$ ,

$$\begin{aligned} \langle V^* \zeta_m \otimes e, h \rangle &= \langle \zeta_m \otimes e, V h \rangle \\ &= \langle e, R T_m^* h \rangle \\ &= \langle T_m R^* e, h \rangle. \end{aligned}$$

Hence  $V^* \zeta_m \otimes e = T_m R^* e$ .

Next note that, from the computation above, equation (8) holds for Laurent polynomials (finite linear combinations of  $\{\zeta_j : j \in \mathbb{Z}\}$ ). Next, if  $f \in H^2(k)$ , then there is a sequence of Laurent polynomials  $p_n$  which converge to  $f$  in  $H^2(k)$  and also uniformly on compact subsets of  $\mathbb{A}$ . Hence,  $p_n \otimes h$  converges in  $H^2(k) \otimes \mathcal{H}$  to  $f \otimes h$  and also  $T_{p_n}$  converges to  $T_f$  in norm, and equation (8) is proved.

Next, if both  $f, g \in H^\infty$ , then

$$\begin{aligned} V^* M_f g \otimes e &= V^* f g 1 \otimes e \\ &= T_{fg} R^* e \\ &= T_f T_g R^* e \\ &= T_f V^* g \otimes e. \end{aligned}$$

Thus,  $V^* M = T V^*$  so that  $M$  lifts  $T$ .  $\square$

**Proposition 3.2.** *Suppose  $T \in \mathcal{B}(\mathcal{M})$  has spectrum in  $\mathbb{A}$ . If  $\frac{1}{k}(T, T^*) \succeq 0$  and if  $R \in \mathcal{B}(\mathcal{M}, \mathcal{H})$  satisfies  $R^* R = \frac{1}{k}(T, T^*)$ , then the sum in equation (7) converges WOT to the identity. In particular,  $M$  lifts  $T$ .*

*Conversely, if  $G$  is a positive operator and the sum*

$$\sum T_n G T_n^*$$

*converges WOT to the identity, then  $G = \frac{1}{k}(T, T^*)$ .*

**Remark 3.3.** It is always possible to choose  $\mathcal{H} = \mathcal{M}$  or  $\mathcal{H} \subset \mathcal{M}$ , though the former choice could lead to a representation which is not minimal.

*Proof.* The first part of the proposition follows from

$$I = k(T, T^*) \left( \frac{1}{k}(T, T^*)(I) \right) = k(T, T^*)(R^* R).$$



The hypothesis for the second part of the lemma is  $k(T, T^*)(G) = I$ . Hence,

$$G = \frac{1}{k}(T, T^*)(k(T, T^*)(G)) = \frac{1}{k}(T, T^*)(I).$$

□

Recall the notion of a minimal lifting given in Definition 2.8.

**Proposition 3.4.** *The lifting  $VT^* = M^*V$  of  $T$  on  $H^2(k) \otimes \mathcal{H}$  is minimal if and only if there does not exist a proper subspace  $\mathcal{F} \subset \mathcal{H}$  such that the range of  $V$  lies in  $H^2(k) \otimes \mathcal{F}$ .*

*Proof.* From the form of  $V$ , the smallest subspace  $\mathcal{F}$  of  $\mathcal{H}$  such that the range of  $V$  lies in  $H^2(k) \otimes \mathcal{F}$  is the closure of the range of  $R = Q^*V$ . □

**Proposition 3.5.** *Suppose  $T \in \mathcal{B}(\mathcal{M})$ . If  $\sigma(T) \subset \mathbb{A}$  and  $\frac{1}{k}(T, T^*) \succeq 0$ , then  $M$  lifts  $T$ .*

*If both  $V_j T^* = M^* V_j$  where  $M$  is acting on  $H^2(k) \otimes \mathcal{H}_j$ ,  $j = 1, 2$  are minimal liftings of  $T$ , then there is a unitary operator  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  so that  $(I \otimes U)V_1 = V_2$ ; i.e., a minimal lifting is unique up to unitary equivalence.*

*Proof.* The first part follows from Proposition 3.2.

From Proposition 3.1,

$$V_\ell h = \sum \zeta_j \otimes R_\ell T_j^* h,$$

where  $R_\ell = Q_\ell^* V_\ell : \mathcal{M} \rightarrow \mathcal{H}_\ell$  and  $Q_\ell^* \sum \zeta_j \otimes f_j = f_0$  on  $H^2(k) \otimes \mathcal{H}_\ell$ . Moreover,

$$I = k(T, T^*)(R_\ell^* R_\ell) = \sum T_j R_\ell^* R_\ell T_j^*.$$

Therefore, by Proposition 3.2  $R_\ell^* R_\ell = \frac{1}{k}(T, T^*)$  for  $\ell = 1, 2$ .

From minimality,  $R_\ell$  has dense range and therefore there is a unitary operator  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  so that  $R_2 = U R_1$ . It follows that  $(I \otimes U)V_1 = V_2$ . □

#### 4. SOME FUNCTIONAL HILBERT SPACES

Theorem 2.9 involves operator-valued measures and implicitly certain related functional Hilbert spaces. In this section we sketch out the relevant constructions. Most of what is needed is summarized later as Lemma 6.1 in Section 6.

**4.1. General constructions.** Let  $\mathfrak{B}(\mathbb{T})$  denote the Borel subsets of the unit circle  $\mathbb{T}$ . By an *operator-valued measure* on  $\mathbb{T}$  we mean a Hilbert space  $\mathcal{M}$  and a function

$$\nu : \mathfrak{B}(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{M})$$

such that

- (p)  $\nu(\omega) \succeq 0$  for  $\omega \in \mathfrak{B}(\mathbb{T})$ ; and
- (ca) for each  $e, f \in \mathcal{M}$ , the function

$$\omega \mapsto \langle \nu(\omega)e, f \rangle$$

is a (complex) measure on  $\mathfrak{B}(\mathbb{T})$ .

A (measurable) partition  $P$  of  $\mathbb{T}$  is a finite disjoint collection  $\omega_1, \dots, \omega_n \in \mathfrak{B}(\mathbb{T})$  whose union is  $\mathbb{T}$ . A measurable simple function  $H$  is a function of the form

$$H = \sum_{j=1}^n K_{\omega_j} c_j$$

for some vectors  $c_j \in \mathcal{M}$  and partition  $P$ . Here,  $K_\omega$  denotes the characteristic function of a set  $\omega$ . Let  $\mathcal{S}$  denote the collection of measurable simple functions.

The measure  $\nu$  gives rise to a semi-inner product on  $\mathcal{S}$  as follows. If  $H' = \sum_{\ell=1}^m K_{\omega'_\ell} c'_\ell$  is also in  $\mathcal{S}$ , define

$$\langle H', H \rangle_\nu = \sum_{j,\ell} c_j^* \nu(\omega_j \cap \omega'_\ell) c'_\ell.$$

In the usual way, this inner product gives rise to a semi-norm,

$$\|H\|_\nu^2 = \langle H, H \rangle_\nu.$$

A *tagging*  $S$  of the partition  $P$  consists of a choice of points  $S = (s_j \in \omega_j)$ . The pair  $(P, S)$  is a *tagged partition*. The collection of tagged partitions is a directed set under the relation  $(P, S) \preceq (Q, T)$  if  $Q$  is a refinement of  $P$ . Given  $F : \mathbb{T} \rightarrow \mathcal{M}$ , let  $F(P, S)$  denote the resulting measurable simple function

$$F(P, S) = \sum K_{\omega_j} F(s_j).$$

Thus, each such  $F$  generates the net  $\{F(P, S) : (P, S)\}$  of simple functions.

Let  $\mathcal{R}^2(\nu)$  denote those  $F$  for which the net  $\{F(P, S)\}$  is bounded and Cauchy in  $\mathcal{S}$ ; i.e., those  $F$  for which there is a  $C$  such that  $\|F(P, S)\|_\nu \leq C$  for all  $(P, S)$ , and such that for each  $\epsilon > 0$  there is a partition  $Q$  such that for any pair  $(P, S), (P', S')$  such that  $P$  and  $P'$  both refine  $Q$ ,

$$(9) \quad \epsilon^2 > \|F(P, S) - F(P', S')\|_\nu^2 = \sum_{j,k} (F(s_j) - F(s'_k))^* \nu(\omega_j \cap \omega'_k) (F(s_j) - F(s'_k)).$$

The following are some simple initial observation.

**Lemma 4.1.** *Measurable simple functions are in  $R^2(\nu)$ .*

*If  $F \in R^2(\nu)$  and  $H \in \mathcal{S}$ , then the net  $\langle H, F(P, S) \rangle_\nu$  is Cauchy.*

*If  $F, G \in R^2(\nu)$ , then the net  $\langle F(P, S), G(P, S) \rangle_\nu$  converges.*

*Proof.* The first statement is evident.

Given tagged partitions  $(P, S)$  and  $(Q, T)$ ,

$$\begin{aligned} & |\langle F(P, S), G(P, S) \rangle_\nu - \langle F(Q, T), G(Q, T) \rangle_\nu| \\ & \leq |\langle F(P, S) - F(Q, T), G(P, S) \rangle_\nu| + |\langle F(Q, T), G(P, S) - G(Q, T) \rangle_\nu|. \end{aligned}$$

This estimate, Cauchy-Schwarz, plus the boundedness hypothesis on the nets proves the third statement.

The second statement is a special case of the third.  $\square$

**Lemma 4.2.** *If  $F, G \in R^2(\nu)$ , then so is  $F + G$ .*

*Proof.* The boundedness of the net  $\{(F + G)(P, S)\}$  is evident. Given tagged partitions  $(P, S)$  and  $(P', S')$ , note that

$$\begin{aligned} & \|(F + G)(P, S) - (F + G)(P', S')\|_\nu \\ & \leq \|F(P, S) - F(P', S')\|_\nu + \|G(P, S) - G(P', S')\|_\nu. \end{aligned}$$

Applying this estimate to appropriate partitions and common refinement proves the result.  $\square$

The assignment,

$$\langle F, G \rangle_\nu = \lim \langle F(P, S), G(P, S) \rangle_\nu$$

defines a semi-inner product on  $\mathcal{R}^2(\nu)$  which is also natural to write as

$$(10) \quad \langle F, G \rangle_\nu = \int \langle d\nu(s)F(s), G(s) \rangle.$$

We define  $L^2(\nu)$  as the completion, after moding out null vectors, of  $\mathcal{R}^2(\nu)$  in the (semi-)norm induced by this (semi-)inner product.

**Proposition 4.3.** *Simple functions are dense in  $L^2(\nu)$ . In particular, the inclusion  $\mathcal{M} \rightarrow L^2(\nu)$  which sends  $m \in \mathcal{M}$  to the equivalence class of the constant function  $m$  is bounded.*

Moreover, if  $H = \sum_1^n K_{\omega_j} m_j$  and  $H' = \sum_1^{n'} K_{\omega'_j} m'_j$ , then

$$\langle H, H' \rangle_\nu = \sum_{j, \ell} \langle \nu(\omega_j \cap \omega'_\ell) m_j, m'_\ell \rangle.$$

*Proof.* Let  $F \in \mathcal{R}^2(\nu)$  and  $\epsilon > 0$  be given. Choose a partition  $Q$  such that for all for all tagged partitions  $(P, S), (P', S')$ , such that  $P$  and  $P'$  refine  $Q$ , the inequality (9) holds. Let  $H = F(Q, T)$ . Then,

$$\epsilon^2 > \|(H - F)(P, S)\|_\nu^2 = \langle H, H \rangle_\nu - \langle H, F(P, S) \rangle_\nu - \langle F(P, S), H \rangle_\nu + \langle F(P, S), F(P, S) \rangle_\nu.$$

In view of Lemma 4.1, the right hand side converges to  $\|H - F\|_\nu^2$  and so (measurable) simple functions are dense in  $\mathcal{R}^2(\nu)$ . Since  $\mathcal{R}^2(\nu)$  is dense in  $L^2(\nu)$  the first statement follows.

The second statement is a restatement of the definition of the inner product induced by  $\nu$  on measurable simple functions.  $\square$

While there is no reason to believe a given continuous  $\mathcal{M}$  valued function on  $\mathbb{T}$  should be in  $L^2(\nu)$ , there is an important class which is.

**Proposition 4.4.** *Suppose  $f : \mathbb{T} \rightarrow \mathcal{B}(\mathcal{M})$  is continuous and  $C$  is a non-negative real number. If, for each  $s$  and  $t$  and Borel set  $\omega$ , both*

$$(11) \quad f(s)\nu(\omega)f(s)^* \leq C\nu(\omega)$$

and

$$(12) \quad (f(s) - f(t))\nu(\omega)(f(s) - f(t))^* \preceq \|f(s) - f(t)\|^2 \nu(\omega),$$

then for each  $m \in \mathcal{M}$ , the function  $f(s)m$  is in  $L^2(\nu)$ .

*Proof.* Fix a vector  $m$  and let  $F(s) = f(s)^*m$ . The inequality of equation (11) implies the net  $\{F(P, S)\}$  is bounded. A straightforward argument using the uniform continuity of  $f$  and the inequality (12) shows that the net  $\{F(P, S)\}$  is Cauchy. Hence  $F \in \mathcal{R}^2(\nu)$ .  $\square$

The algebra  $C(\mathbb{T})$  of continuous (scalar-valued) functions on  $\mathbb{T}$  has a natural representation on  $L^2(\nu)$ .

**Lemma 4.5.** *If  $a \in C(\mathbb{T})$  and  $F \in \mathcal{R}^2(\nu)$ , then  $aF \in \mathcal{R}^2(\nu)$  and moreover,  $\|aF\|_\nu \leq \|a\|_\infty \|F\|_\nu$ . Hence  $a$  determines a bounded linear operator  $\tau(a)$  on  $L^2(\nu)$ . The mapping sending  $a \in C(\mathbb{T})$  to  $\tau(a) \in \mathcal{B}(L^2(\nu))$  is a unital  $*$ -representation.*

Finally, given  $a, a' \in C(\mathbb{T})$  and simple measurable functions  $F = \sum K_{\omega_j} m_j$  and  $F' = \sum K_{\omega'_\ell} m'_\ell$ ,

$$(13) \quad \langle a' F', a F \rangle_\nu = \sum_{j, \ell} \int_{\omega_j \cap \omega'_\ell} a'(s) a^*(s) \langle d\nu(s) m'_\ell, m_j \rangle.$$

*Proof.* Fix  $F \in \mathcal{R}^2(\nu)$ . For any partition  $P = (\omega_j)$  of  $\mathbb{T}$  and pointing  $S = (s_j \in \omega_j)$ ,

$$\begin{aligned} \|(aF)(P, S)\|_\nu^2 &= \sum \langle \nu(\omega_j) a(s_j) F(s_j), a(s_j) F(s_j) \rangle \\ &= \sum |a(s_j)|^2 \langle \nu(\omega_j \cap \omega) F(s_j), F(s_j) \rangle \\ &\leq \|a\|_\infty^2 \|F(P, S)\|_\nu^2. \end{aligned}$$

Thus, since the net  $\{F(P, S)\}$  is bounded, so is the net  $\{aF(P, S)\}$ .

If  $(R, T)$  is another tagged partition, where  $R = (\theta_\ell)$  and  $T = (t_\ell \in \theta_\ell)$ , then  $(aF)(P, S) - (aF)(R, T) = G + H$ , where

$$\begin{aligned} G &= \sum_{j, \ell} (a(s_j) - a(t_\ell)) K_{\omega_j \cap \theta_\ell} F(s_j), \\ H &= \sum_{j, \ell} a(t_\ell) K_{\omega_j \cap \theta_\ell} (F(s_j) - F(t_\ell)). \end{aligned}$$

If  $\epsilon$  bounds both  $|a(s_j) - a(t_\ell)|$  and  $\|F(P, S) - F(R, T)\|_\nu$  and if  $C$  is a bound for the net  $\{F(P, S)\}$ , then

$$\|G\|_\nu \leq \epsilon C, \quad \|H\|_\nu \leq \|a\|_\infty \epsilon.$$

Thus, using the uniform continuity of  $a$  and the fact that the net  $\{F(P, S)\}$  is Cauchy, it is possible to choose a partition  $Q$  of sufficiently small width so that if  $P$  and  $R$  are refinements of  $Q$  with taggings  $S$  and  $T$  respectively, then

$$\begin{aligned} \|(aF)(P, S) - (aF)(R, T)\|_\nu &= \|G + H\|_\nu \\ &\leq \|G\|_\nu + \|H\|_\nu \leq (C + \|a\|_\infty) \epsilon. \end{aligned}$$

Thus the net  $\{(aF)(P, S)\}$  is Cauchy. Hence  $aF \in \mathcal{R}^2(\nu)$ .

It suffices to prove equation (13) in the case that  $F = K_\omega m$  and  $F' = K_{\omega'} m'$ . Given a partition  $(P, S)$ ,

$$\begin{aligned} \langle (a' F')(P, S), (aF)(P, S) \rangle_\nu &= \sum a'(s_j)^* a(s_j) \langle \nu(\omega_j \cap \omega \cap \omega') m' m \rangle \\ &= \int_{\omega \cap \omega'} \sum a'(s_j)^* a(s_j) K_{\omega_j} \langle d\nu(s) m', m \rangle. \end{aligned}$$

Given  $\epsilon > 0$ , if the partition  $P$  is chosen, using the uniform continuity of  $a' a^*$ , so that

$$\| [a' a^* - \sum_j a'(s_j) a^*(s_j) K_{\omega_j}] K_{\omega \cap \omega'} \|_\infty < \epsilon,$$

then

$$\left| \int_{\omega \cap \omega'} [a' a^* - \sum (a'(s_j))^* a(s_j) K_{\omega_j}] \langle d\nu(s) m', m \rangle \right| \leq \epsilon \|\nu(\omega \cap \omega')\| \|m'\| \|m\|.$$

It follows that the net  $\langle (a' F')(P, S), (aF)(P, S) \rangle_\nu$  converges to the integral

$$\int_{\omega \cap \omega'} a'(s) a^*(s) \langle d\nu(s) m', m \rangle,$$

completing the proof of equation (13).

Each  $a$  determines a bounded operator on  $R^2(\nu)$  (with norm at most  $\|a\|_\infty$ ) and hence extends to a bounded operator  $\tau(a)$  on all of  $L^2(\nu)$ . It remains to prove that  $\tau$  determines a unital  $*$ -representation on  $L^2(\nu)$ . Evidently  $\tau(1) = I$ . Using equation (13) twice (first with  $a = 1$  and the second with  $a = (a')^*$  and  $a' = 1$ ),

$$\begin{aligned} \langle \tau(a')^* F, F' \rangle_\nu &= \langle F, \tau(a') F \rangle_\nu \\ &= \langle F, a' F' \rangle_\nu \\ &= \int (a')^*(s) \langle d\mu(s) m, m' \rangle \\ &= \langle \tau((a')^*) F, F' \rangle_\nu. \end{aligned}$$

Hence  $\tau(a)^* = \tau(a^*)$ .

Finally, again using equation (13) twice, this time first with  $a = aa'$   $a' = 1$ , and second with  $a = a$  and  $a' = (a')^*$ ,

$$\begin{aligned} \langle \tau(aa') F, F' \rangle_\nu &= \int a' a(s) \langle d\mu(s) m, m' \rangle_\nu \\ &= \int ((a')^*)^* a(s) \langle d\mu(s) m, m' \rangle \\ &= \langle \tau(a')^* \tau(a) F, F' \rangle_\nu \\ &= \langle \tau(a') \tau(a) F, F' \rangle_\nu. \end{aligned}$$

Thus  $\tau(a'a) = \tau(a')\tau(a)$ . □

**4.2. Agler decompositions again.** Suppose  $\mu$  is an Agler decomposition as defined in subsection 2.6. Then both  $\mu$ , and  $\Lambda$  defined by

$$\Lambda(\omega) = k(T, T^*)(\mu(\omega)),$$

are positive  $\mathcal{B}(\mathcal{M})$ -valued measures on  $\mathfrak{B}(\mathbb{T})$  and the constructions of the previous section apply to both  $L^2(\mu)$  and  $L^2(\Lambda)$ .

**Lemma 4.6.** *If  $F \in \mathcal{R}^2(\Lambda)$ , then  $F \in \mathcal{R}^2(\mu)$  and  $\langle F, F \rangle_\Lambda \geq \langle F, F \rangle_\mu$ . Thus, the mapping  $\Phi^* : \mathcal{R}^2(\Lambda) \rightarrow \mathcal{R}^2(\mu)$  given by  $F \mapsto F$  induces a contractive linear mapping  $\Phi^* : L^2(\Lambda) \rightarrow L^2(\mu)$ .*

*Proof.* This follows immediately from

$$\Lambda(\omega) = k(T, T^*)(\mu(\omega)) \succeq \mu(\omega).$$

□

Given  $m \in \mathcal{M}$ , let  $Y$  denote the mapping  $Y : \mathcal{M} \rightarrow L^2(\Lambda)$  defined by  $Ym(\psi) = T_\psi^* m$ . Here the identification of  $\Psi$ , the collection of test functions, with  $\mathbb{T}$  is in force. Of course, it needs to be verified that  $Ym(\psi)$  is indeed in  $L^2(\Lambda)$ . Let  $\iota$  denote the inclusion, as constant functions, of  $\mathcal{M}$  into  $R^2(\Lambda)$ . Thus, if  $m \in \mathcal{M}$ , then  $\iota m$  denotes the constant function  $\iota m(\psi) = m$ .

**Lemma 4.7.** *For  $m \in \mathcal{M}$ , the function  $Ym$  is in  $\mathcal{R}^2(\Lambda)$ .*

*Moreover,*

$$\langle \Lambda(\mathbb{T})m, m \rangle_{\mathcal{M}} = \langle \iota m, \iota m \rangle_{L^2(\Lambda)} \geq \langle Ym, Ym \rangle_{L^2(\Lambda)}.$$

Thus,  $Y$  determines a bounded linear operator  $Y : \mathcal{M} \rightarrow L^2(\Lambda)$  given by  $(Ym)(\psi) = T_\psi^* m$ . In the notation of equation (10),

$$(14) \quad \langle Y^* Y m, m \rangle_\Lambda = \left\langle \int d\Lambda(\psi) T_\psi^* m, T_\psi^* m \right\rangle.$$

Further,  $\Phi^* Y : \mathcal{M} \rightarrow L^2(\mu)$  is bounded and

$$(15) \quad \langle Y^* \Phi \Phi^* Y m, m' \rangle_\nu = \int \langle d\mu(\psi) T_\psi^* m, T_\psi^* m' \rangle.$$

**Remark 4.8.** We interpret equations (14) and (15) as

$$(16) \quad Y^* Y = \int T_\psi d\Lambda(\psi) T_\psi^*$$

and

$$(17) \quad Y^* \Phi \Phi^* Y = \int T_\psi d\mu(\psi) T_\psi^*$$

respectively.

Given a tagged partition  $(P, S)$ , let

$$\Delta(P, S, \Lambda) = \sum T_{s_j} \Lambda(\omega_j) T_{s_j}^*$$

and define  $\Delta(P, S, \mu)$  similarly. Thus,  $\Delta(P, S, \Lambda)$  is an operator on  $\mathcal{M}$  and because  $T_s \Lambda(\omega) T_s^* \leq \Lambda(\omega)$ , it is positive semidefinite and bounded above by  $\Lambda(\mathbb{T})$ . For vectors  $m, m' \in \mathcal{M}$ ,

$$\langle \Delta(P, S, \Lambda) m, m' \rangle = \langle Y m(P, S), Y m'(P, S) \rangle_\Lambda.$$

Thus, the net  $\{\Delta(P, S, \Lambda)\}$  converges WOT to the operator of equation (16).

It follows that the net  $\{\frac{1}{k}(T, T^*)(\Delta(P, S, \Lambda))\}$  also converges. On the other hand,

$$\frac{1}{k}(T, T^*)(\Delta(P, S, \Lambda)) = \Delta(P, S, \mu).$$

Hence the net  $\{\Delta(P, S, \mu)\}$  converges WOT to the operator of equation (17).

*Proof.* By hypothesis, for  $\varphi$  in the scalar Schur class and measurable sets  $\omega$ ,

$$(18) \quad \Lambda(\omega) \succeq T_\varphi \Lambda(\omega) T_\varphi^*.$$

Thus, the functions  $Ym$  satisfies the hypotheses, with respect to  $\Lambda$ , of Proposition 4.4. It follows that  $Ym$  is in  $L^2(\Lambda)$  for each  $m$ .

The moreover follows immediately from equation (18).

The rest of the Lemma follows from the definitions.  $\square$

**Lemma 4.9.** Let  $\mu$  be an Agler decomposition of the pair  $(T, X)$  and let, as in Proposition 3.1,  $R^* R = \frac{1}{k}(T, T^*)$ . Then,

$$R^* R + Y^* \Phi \Phi^* Y = X R^* R X^* + \iota^* \Phi \Phi^* \iota.$$

*Proof.* Part (ii) of the definition of an Agler decomposition can be written as

$$\frac{1}{k}(T, T^*)(I) + \int_\Psi T_\psi d\mu(\psi) T_\psi^* = \frac{1}{k}(T, T^*)(X X^*) + \mu(\mathbb{T}).$$

Because  $X$  commutes with  $T^*$ ,

$$\frac{1}{k}(T, T^*)(X X^*) = X \frac{1}{k}(T, T^*)(I) X^*$$

and hence  $\frac{1}{k}(T, T^*)(XX^*) = XR^*RX^*$ . An application of the last part of Lemma 4.7 gives

$$R^*R + Y^*\Phi\Phi^*Y = XR^*RX^* + \mu(\mathbb{T}).$$

Noting that

$$\langle \iota^*\Phi\Phi^*\iota m, m' \rangle = \langle m, m' \rangle_{L^2(\mu)} = \langle \mu(\mathbb{T})m, m' \rangle$$

completes the proof.  $\square$

## 5. UNIFORMITY OF THE TEST FUNCTIONS

Using the orthonormal basis  $\{\zeta_j\}$  for  $H^2(k)$  defined in equation (1), each test function  $\psi$  has a Laurent expansion,

$$\psi = \sum \langle \psi, \zeta_j \rangle \zeta_j.$$

In this section we show that

$$T_\psi^* = \sum \langle \zeta_j, \psi \rangle T_j^*$$

with convergence in the strong operator topology.

The section begins with establishing a uniform, independent of  $\psi$ , estimate on the rate of convergence of the Laurent series for  $\psi$  on compact subsets of  $\mathbb{A}$ .

**Lemma 5.1.** *There is a  $0 < \rho < 1$  and a constant  $C$  so that for all  $\psi \in \Psi$  and  $j \in \mathbb{Z}$ ,*

$$|\langle \psi, \zeta_j \rangle| < C\rho^{|j|}.$$

*Sketch of proof.* There is a function  $\varphi$  analytic in a neighborhood of our annulus  $\mathbb{A}$  such that

- (a) for  $|z| = 1$ ,  $|\varphi(z)| = 1$ ;
- (b) for  $|z| = q$ ,  $|\varphi(z)| = \sqrt{q}$ ; and
- (c)  $\varphi(\sqrt{q}) = 0$ .

It extends by reflection across both boundaries to be analytic in the annulus  $\{q^{\frac{3}{2}} < |z| < q^{-\frac{1}{2}}\}$  (see Section 11).

It follows that, up to a unimodular constant, if  $\psi$  is unimodular on the boundary of  $\mathbb{A}$  and has exactly two zeros, these being at  $\sqrt{q}$  and  $\sqrt{q}\gamma$  (for a necessarily unimodular  $\gamma$ ), then

$$(19) \quad \psi(z) = \delta \frac{\varphi(z)\varphi(\gamma^*z)}{z},$$

for some unimodular  $\delta$ . In particular equation (19) gives an explicit parametrization of  $\Psi$  by  $\mathbb{T}$ .

It now follows that  $\psi \in \Psi$  is bounded uniformly (independent of  $\psi$ ) on a larger annulus than  $\mathbb{A}$  and the result follows.  $\square$

In the following Lemma  $\mu$  is an Agler decomposition for  $(T, X)$ . Thus,  $\Lambda(\omega) = k(T, T^*)(\mu(\omega))$  and for  $\varphi$  in the scalar Schur class,  $T_\varphi\Lambda(\omega)T_\varphi^* \preceq \Lambda(\omega)$ .

**Lemma 5.2.** *If  $m \in \mathcal{M}$ , then, for each  $j$ , the function  $\langle \zeta_j, \psi \rangle T_j^* m \in L^2(\Lambda)$  and moreover, independent of  $j$ , there is a  $C > 0$  and  $0 < \rho < 1$  such that*

$$\|\langle \zeta_j, \psi \rangle T_j^* m\|_{L^2(\Lambda)} \leq C\rho^{|j|}.$$

If  $G \in L^2(\Lambda)$  and  $m \in \mathcal{M}$ , then

$$(20) \quad \langle G, Ym \rangle_{L^2(\Lambda)} = \sum_j \langle G, \langle \zeta_j, \psi \rangle T_j^* m \rangle_{L^2(\Lambda)}.$$

If  $F$  is a measurable simple function, then

$$\langle F, \Phi^* Ym \rangle_{L^2(\mu)} = \sum_j \langle F, \langle \zeta_j, \psi \rangle T_j^* m \rangle_{L^2(\mu)}.$$

*Proof.* Given a positive integer  $N$ , define  $\sigma_N : \mathbb{T} \rightarrow H^\infty(\mathbb{A})$  by

$$\sigma_N(\psi) = \sum_{|j| \leq N} \langle \zeta_j, \psi \rangle \zeta_j.$$

In view of Lemma 5.1, the sequence  $\sigma_N$  converges to the identity function  $\psi$  uniformly on compact subsets of  $\mathbb{A}$ . Hence, by Proposition 4.4, for each  $m \in \mathcal{M}$

$$\|T_{\psi - \sigma_N}^* m\|_{L^2(\Lambda)} = \|T_\psi^* m - \sum_{|j| \leq N} \langle \zeta_j, \psi \rangle T_j^* m\|_{L^2(\Lambda)}$$

converges to 0 and equation (20) follows.

To finish the proof, choose  $G = \Phi F$  in equation (20) to obtain

$$\begin{aligned} \langle \Phi F, Ym \rangle_{L^2(\Lambda)} &= \sum_j \langle \Phi F, \langle \zeta_j, \psi \rangle T_j^* m \rangle_{L^2(\Lambda)} \\ &= \sum_j \langle F, \langle \zeta_j, \psi \rangle T_j^* m \rangle_{L^2(\mu)}. \end{aligned}$$

using that  $\Phi^* : L^2(\Lambda) \rightarrow L^2(\mu)$  is the inclusion mapping (and is bounded). The final conclusion of the lemma follows.  $\square$

## 6. THE FACTORIZATION AND LURKING ISOMETRY

The next several sections, Sections 6, 7, and 8, are devoted to the proof of (ad) implies (sc) in Theorem 2.9 and throughout these sections the relevant hypotheses are in force. Namely,  $\mathcal{M}$  is a separable Hilbert space,

- (a)  $X, T \in \mathcal{B}(\mathcal{M})$  commute;
- (b)  $\sigma(T) \subset \mathbb{A}$ ;
- (c)  $T$  lifts to  $M$  on  $H^2(k) \otimes \mathcal{M}$  via  $VT^* = M^*V$  and

$$Vh = \sum \zeta_j \otimes RT_j^* h,$$

where  $R^*R = \frac{1}{k}(T, T^*)$ ; and

- (d) there exists a measure  $\mu : \mathfrak{B}(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{M})$  such that, with  $\Lambda(\omega) = k(T, T^*)(\mu(\omega))$ ,

$$\Lambda(\omega) - T_\varphi \Lambda(\omega) T_\varphi^* \succeq 0$$

for all Borel subset  $\omega$  and Schur class functions  $\varphi$  and

$$\frac{1}{k}(T, T^*)(I - XX^*) = \mu(\mathbb{T}) - \int T_\psi d\mu(\psi) T_\psi^*.$$

Once properly formulated to account for infinitely many test functions, the overarching strategy for proving results like Theorem 2.9 is now well established, but the presence of infinitely many, and not necessarily orthogonal, test functions requires some reinterpretation of earlier results, revealing new structures. The positivity condition in (ad) (item (d) above) is *factored* and this factorization produces a



*lurking isometry* and of course an auxiliary Hilbert space. The lurking isometry in turn generates the  $\Psi$ -unitary colligation. A good deal of effort is required to show that the resulting transfer function solves the problem and the argument given here is patterned after that in [29], which in turn borrowed from [14] [11] and closely related to those in [3].

The factorization we will need comes from *factoring* the measure  $\Lambda$  of Remark 2.7. This factorization amounts to the construction of the Hilbert spaces  $L^2(\Lambda)$  and  $L^2(\mu)$  in Section 4. The following Lemma summarizes many of the needed results and constructions from Section 4

**Lemma 6.1.** *With the hypotheses above,*

- (i) *there exist Hilbert spaces  $L^2(\Lambda)$  and  $L^2(\mu)$  which contain densely all simple measurable  $\mathcal{M}$ -valued functions so that, in particular, the inclusion mapping  $\iota : \mathcal{M} \rightarrow L^2(\Lambda)$  is bounded (not necessarily isometric);*
- (ii) *the space  $L^2(\Lambda)$  includes in  $L^2(\mu)$  contractively so that there exists an operator  $\Phi$  whose adjoint  $\Phi^* : L^2(\Lambda) \rightarrow L^2(\mu)$  is the inclusion mapping; and*
- (iii) *an operator  $Y : \mathcal{M} \rightarrow L^2(\Lambda)$  defined by  $Ym = T_\psi^* m$  (that is, the function  $Ym(\psi) = T_\psi^* m$  determines an element of  $L^2(\Lambda)$ ),*

*which together satisfy the lurking isometry equality,*

$$(21) \quad R^*R + Y^*\Phi\Phi^*Y = XR^*RX^* + \iota^*\Phi\Phi^*\iota.$$

*Moreover, if*

- (a)  $a, a' : \mathbb{T} \rightarrow \mathbb{C}$  *are continuous;*
- (b)  $\omega, \omega'$  *are Borel subsets of  $\mathbb{T}$ ;*
- (c)  $m, m' \in \mathcal{M}$ ; *and*
- (d)  $F = K_\omega m$  *and*  $F' = K_{\omega'} m'$ ,

*then  $aF$  and  $a'F'$  are in  $L^2(\mu)$  and*

$$\langle aF, a'F' \rangle = \int_{\omega \cap \omega'} a(\psi) a'(\psi)^* \langle d\mu(\psi) m, m' \rangle.$$

*In particular, if  $F \in L^2(\mu)$  is simple, then  $aF$  determines an element of  $L^2(\mu)$  and  $\|aF\| \leq \|a\|_\infty \|F\|$ . Thus, there is a unital  $*$ -representation  $\tau : C(\mathbb{T}) \rightarrow \mathcal{B}(L^2(\mu))$  such that  $\tau(E(z))F(\psi) = \psi(z)F(\psi)$ . (Recall the identification of  $\Psi$ , the collection of test functions, with  $\mathbb{T}$ .)*

Condition (i) in the definition of Agler decomposition implies  $Y$  is a bounded (in fact contractive) operator into  $L^2(\Lambda)$ . (Details in Section 4). Further,  $\Phi^*Ym = T_\psi^* m$  determines an element of  $L^2(\mu)$  and in condition (ii) in the definition of an Agler decomposition equation (5) becomes,

$$\langle \Phi^*Ym, \Phi^*Ym \rangle = \int \langle T_\psi d\mu(\psi) T_\psi^* m, m \rangle.$$

Thus

$$\frac{1}{k}(T, T^*) - X \frac{1}{k}(T, T^*) X^* = \iota^* \Phi \Phi^* \iota - Y^* \Phi \Phi^* Y.$$

Rearranging and using the relation  $\frac{1}{k}(T, T^*) = R^*R$  of Proposition 3.2 produces the lurking isometry equality of equation (21).

## 7. THE COLLIGATION AND ITS TRANSFER FUNCTION

Recall there are two parts to the colligation. The unitary matrix and the representation.

**7.1. The unitary matrix.** The lurking isometry, equation (21), produces, non-uniquely, the unitary matrix of item (iii) of Definition 2.1. The construction requires an initial enlargement of the space  $L^2(\mu)$ . Let  $\ell^2$  denote the usual separable Hilbert space with orthonormal basis  $\{e_j : j \in \mathbb{N}\}$  and define  $\mathcal{W} : L^2(\mu) \rightarrow L^2(\mu) \otimes \ell^2$  by  $\mathcal{W}F = F \otimes e_0$ . In particular,  $\mathcal{W}$  is an isometry.

Let  $\mathcal{K}$  and  $\mathcal{K}_*$  denote the subspaces of  $[L^2(\mu) \otimes \ell^2] \oplus \mathcal{H}$  given by the closures of the spans of

$$\{(\mathcal{W}\Phi^*Ym \oplus Rm) : m \in \mathcal{M}\}, \quad \{(\mathcal{W}\Phi^*\iota m \oplus RX^*m) : m \in \mathcal{M}\}$$

respectively, where  $\iota$  is the inclusion of  $\mathcal{M}$  into  $L^2(\Lambda)$ . The lurking isometry of equation (21) says that the mapping from  $\mathcal{K}$  to  $\mathcal{K}_*$  defined by

$$(\mathcal{W}\Phi^*Ym \oplus Rm) \rightarrow (\mathcal{W}\Phi^*\iota m \oplus RX^*m)$$

is an isometry. Because  $\mathcal{K}$  and  $\mathcal{K}_*$  have the same codimension (i.e., their orthogonal complements have the same dimension), this isometry can be extended to a unitary

$$U = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} : \begin{array}{c} L^2(\mu) \otimes \ell^2 \\ \oplus \\ \mathcal{H} \end{array} \rightarrow \begin{array}{c} L^2(\mu) \otimes \ell^2 \\ \oplus \\ \mathcal{H} \end{array},$$

giving rise to the usual system of equations,

$$(22) \quad \begin{aligned} A^*\mathcal{W}\Phi^*Y + C^*R &= \mathcal{W}\Phi^*\iota \\ B^*\mathcal{W}\Phi^*Y + D^*R &= RX^*. \end{aligned}$$

Note that the domain of  $D$  and  $B$  and the codomain of  $C$  is  $\mathcal{M}$ .

**7.2. The representation.** Of course we also need the representation  $\rho : C(\mathbb{T}) \rightarrow \mathcal{B}(L^2(\mu) \otimes \ell^2)$  of item (ii) in Definition 2.1. We begin with the unital representation  $\tau : C(\mathbb{T}) \rightarrow \mathcal{B}(L^2(\mu))$  from Lemma 6.1 (see also Lemma 4.5) and define  $\rho = \tau \otimes I$ , where  $I$  is the identity on  $\ell^2$ .

**7.3. The transfer function and its properties.** Let  $E(z) : \Psi \rightarrow \mathbb{C}$  denote evaluation at  $z$ ; i.e.,  $E(z)(\psi) = \psi(z)$ . For  $F \in L^2(\mu)$ ,  $\tau(E(z))F(\psi) = \psi(z)F(\psi)$ . The corresponding transfer function is then given by

$$(23) \quad W(z) = D + C(I - \rho(E(z))A)^{-1}\rho(E(z))B.$$

The function  $W$  gives rise to the multiplication operator  $M_W$  on  $H^2(k) \otimes \mathcal{M}$ . In the following subsection we make some observations related to  $M_W$  and the corresponding  $\Psi$ -unitary colligation needed in the sequel.

There is a canonical auxiliary multiplication operator associated to  $z \mapsto \rho(E(z))$  which, as in equation (3), is most conveniently defined in terms of its adjoint. Define  $Z^* : H^2(k) \otimes [L^2(\mu) \otimes \ell^2] \rightarrow H^2(k) \otimes [L^2(\mu) \otimes \ell^2]$  by

$$Z^*(k_z \otimes F) = k_z \otimes \rho(E(z))^*F.$$

Of course it needs to be checked that, after extending by linearity, this prescription produces a bounded operator, a fact that follows readily from

$$\begin{aligned} & \langle k_z \otimes \sum F_j \otimes e_j, k_w \otimes \sum G_\ell \otimes e_\ell \rangle - \sum_j \langle Z^* k_z \otimes F_j, Z^* k_w \otimes G_j \rangle \\ &= \sum_j k(w, z) [\langle F, G \rangle - \langle \rho(E(z))^* F_j, \rho(E(w))^* G_j \rangle] \\ &= \sum_j \int k(w, z) (1 - \psi(z)^* \psi(w)) G(\psi)^* d\mu(\psi) F(\psi) \end{aligned}$$

and the fact that each  $k(z, w)(1 - \psi(z)^* \psi(w)^*)$  is a positive kernel and  $\mu$  is a positive measure. Here we have used Proposition 2.4 and have actually proved that  $Z$  has norm at most one.

Thus  $\rho(E(z))$  determines a (multiplication) operator on  $H^2(k) \otimes [L^2(\mu) \otimes \ell^2]$  denoted by  $Z$ :

$$\langle Z\mathbf{f}, k_z \otimes F \rangle_{H^2(k) \otimes [L^2(\mu) \otimes \ell^2]} = \langle \rho(E(z))\mathbf{f}(z), F \rangle_{L^2(\mu) \otimes \ell^2}.$$

**Lemma 7.1.** *Given a simple measurable function  $F = \sum K_{\omega_\ell} m_\ell \in L^2(\mu)$  and  $f \in H^\infty$ ,*

$$(24) \quad Z(f \otimes (F \otimes e_p)) = \sum f \zeta_j \otimes \langle \psi, \zeta_j \rangle F \otimes e_p.$$

Here  $K_{\omega_\ell}$  is the characteristic function of the Borel set  $\omega \subset \mathbb{T}$ ;  $e_p$  is the element of  $\ell^2$  with a 1 in the  $p$ -th entry and 0 elsewhere; and the symbol  $\psi$  denotes the variable in  $\Psi$ .

In particular, the sum on the right hand side converges. Since  $\langle \psi, \zeta_j \rangle$  is continuous, it follows, from the moreover part of Lemma 6.1 that  $\langle \psi, \zeta_j \rangle K_{\omega_\ell} m_\ell$  is in  $L^2(\mu)$ .

In Section 11 we show that there is a  $0 < \rho < 1$  and a  $C$  such that for all  $j$ ,  $|\langle \psi, \zeta_j \rangle| < C\rho^{|j|}$  (see also Lemma 5.1). Note also,

$$\psi(z) = \sum_j \langle \psi, \zeta_j \rangle \zeta_j(z).$$

*Proof.* Choose  $C$  and  $\rho$  as above. It follows from Lemma 4.5 that

$$\|\langle \psi, \zeta_j \rangle F\|_{L^2(\mu)} \leq C\rho^{|j|} \|F\|$$

and thus the sum on the right hand side of equation (24) converges.

Because simple functions are dense in  $L^2(\mu)$  by item (i) of Lemma 6.1, it suffices to prove the result assuming  $F = K_\omega m \otimes e_p$ , for a Borel set  $\omega$ . Given  $z \in \mathbb{A}$  and a

(very) simple function  $F' = K_{\omega'} m' \otimes e_p$

$$\begin{aligned}
\langle Z(f \otimes F \otimes e_p), k_z \otimes F' \otimes e_p \rangle &= \langle (f \otimes F \otimes e_p), k_z \otimes \rho(E(z))^* (F' \otimes e_p) \rangle_{H^2(k) \otimes [L^2(\mu) \otimes \ell^2]} \\
&= \langle (f \otimes F), k_z \otimes \tau(E(z))^* F' \rangle_{H^2(k) \otimes L^2(\mu)} \\
&= \int_{\omega \cap \omega'} f(z) \psi(z) \langle d\mu(\psi) m, m' \rangle \\
&= \sum_j f(z) \zeta_j(z) \int_{\omega \cap \omega'} \langle \psi, \zeta_j \rangle \langle d\mu(\psi) m, m' \rangle \\
&= \sum_j f(z) \zeta_j(z) \langle \langle \psi, \zeta_j \rangle F, F' \rangle \\
&= \sum_j \langle f \zeta_j \otimes \langle \langle \psi, \zeta_j \rangle F \otimes e_p, k_z \otimes F' \otimes e_p \rangle.
\end{aligned}$$

□

Returning to the transfer function  $W$  of equation (23), let  $\mathbb{W} = W(z) - D$ . Before concluding this subsection, we present two key relations amongst  $V, \mathbb{W}, R, \Phi, \iota$  and  $Z$ . Define  $J : \mathcal{M} \rightarrow H^2(k) \otimes \mathcal{M}$  by

$$Jm = \sum \zeta_j \otimes T_j^* m.$$

The spectral condition  $\sigma(T) \subset \mathbb{A}$  implies this sum converges and  $J$  is a bounded operator. Note that  $(I \otimes R)J = V$ .

**Lemma 7.2.** *For  $f \in H^\infty(\mathbb{A})$  and  $F \in L^2(\mu) \otimes \ell^2$ ,*

$$(25) \quad J^*(I \otimes \iota^* \Phi \mathcal{W}^*) Z f \otimes F = T_f Y^* \Phi \mathcal{W}^* F,$$

and

$$J^*(I \otimes \iota^* \Phi \mathcal{W}^*) f \otimes F = T_f \iota^* \Phi \mathcal{W}^* F.$$

*Proof.* First, suppose  $f = \zeta^p$  for some integer  $p$ . Straightforward computation and the fact that  $M$  lifts  $T$  gives,

$$\langle \zeta^p \zeta_\ell, \zeta_{p+\ell} \rangle T_{p+\ell} = T_\ell T^p.$$

Given  $m \in \mathcal{M}$ ,  $\omega \in \mathfrak{B}(\mathbb{T})$  and  $h \in L^2(\mu) \otimes \ell^2$ , let  $F = K_\omega \otimes h$ , and compute,

$$\begin{aligned}
&\langle J^*(I \otimes \iota^* \Phi \mathcal{W}^*) Z(z^p \otimes F), m \rangle_{\mathcal{M}} \\
&= \langle (I \otimes \iota^* \Phi \mathcal{W}^*) Z(z^p \otimes F), \sum \zeta_j \otimes T_j^* m \rangle_{H^2(k) \otimes \mathcal{M}} \\
&= \sum_j \langle I \otimes \mathcal{W}^* Z(z^p \otimes F), \zeta_j \otimes \Phi^* \iota T_j^* m \rangle_{H^2(k) \otimes L^2(\mu)} \\
&= \sum_\ell \langle [\sum_\ell z^p \zeta_\ell \otimes \langle \psi, \zeta_\ell \rangle \mathcal{W}^* F], \zeta_j \otimes \Phi^* \iota T_j^* m \rangle_{H^2(k) \otimes L^2(\mu)} \\
&= \sum_\ell \langle \zeta^p \zeta_\ell, \zeta_{p+\ell} \rangle \langle \psi, \zeta_\ell \rangle \langle \mathcal{W}^* F, \Phi^* \iota T_{p+\ell}^* m \rangle_{L^2(\mu)} \\
&= \sum_\ell \langle \Phi \mathcal{W}^* F, \langle \zeta_\ell, \psi \rangle T_\ell^* (T^*)^p m \rangle_{L^2(\Lambda)} \\
&= \langle \Phi \mathcal{W}^* F, Y (T^*)^p m \rangle_{L^2(\Lambda)} \\
&= \langle T^p Y^* \Phi \mathcal{W}^* F, m \rangle_{\mathcal{M}}.
\end{aligned}$$

Here we have used the form of  $V$  from Proposition 3.1 in the second equality; the description of  $Z$  provided by Lemma 7.1 in the fourth; and Lemma 5.2, equation (20) in the seventh.

Now use linearity and the fact that the linear span of elements like  $F$  is dense in  $L^2(\mu) \otimes \ell^2$  to finish the proof of the first part of Lemma 7.2.

An argument very much like the one that proved the first identity proves the second.  $\square$

We now use Lemma 7.2 to establish the following Lemma.

**Lemma 7.3.** *With notations as above (and  $A$ ,  $B$  and  $C$  appearing in the representation of the transfer function  $W$ ),*

$$J^*(I \otimes \iota^* \Phi \mathcal{W}^*)[I - Z(I \otimes A)] = V^*(I \otimes C).$$

*Proof.* For  $f \in H^\infty(\mathbb{A})$  and  $F \in L^2(\mu) \otimes e_p$ ,

$$\begin{aligned} J^*(I \otimes \iota^* \Phi \mathcal{W}^*)[I - Z(I \otimes A)](f \otimes F) &= J^*(I \otimes \iota^* \Phi \mathcal{W}^*)[f \otimes F - Z(f \otimes AF)] \\ &= T_f[\iota^* \Phi \mathcal{W}^* - Y^* \Phi \mathcal{W}^* A]F \\ &= T_f[R^* C]F \\ &= V^*[f \otimes CF]. \end{aligned}$$

Here both parts of Lemma 7.2 were used in the second equality, equation (22) (i) was used in the third, and Proposition 3.1 in the last.

Since the linear span of elements of the form  $f \otimes F$  is dense in  $H^2(k) \otimes [L^2(\mu) \otimes \ell^2]$ , the result follows.  $\square$

The following Lemma does the heavy lifting in the proof of (ad) implies (sc) in Theorem 2.9. Recall  $\mathbb{W} = W - D$ .

**Lemma 7.4.** *For  $m \in \mathcal{M}$ ,*

$$J^*(I \otimes \iota^* \Phi \mathcal{W}^*)Z(I \otimes B)(1 \otimes m) = V^*M_{\mathbb{W}}(1 \otimes m).$$

*Proof.* Choose a sequence  $0 < t_n < 1$  converging to 1 and let

$$\mathcal{Z}_n = (1 - t_n)[I - t_n Z(I \otimes A)]^{-1}.$$

We claim that  $\mathcal{Z}_n$  converges to 0 in the WOT. The first step in proving this claim is to show that  $\mathcal{Z}_n$  is contractive which follows from the following computation in which we have written  $S$  in place of  $Z(I \otimes A)$ :

$$\begin{aligned} I - \mathcal{Z}_n \mathcal{Z}_n^* &= (I - t_n S)^{-1}[(I - t_n S)(I - t_n S)^* - (1 - t_n)^2](I - t_n S)^{-*} \\ &= t_n(I - t_n S)^{-1}[-S - S^* - t_n(1 - SS^*)](I - t_n S)^{-*} \\ &= t_n(I - t_n S)^{-1}[(I - S)(I - S^*) + (1 - t_n)(1 - SS^*)](I - t_n S)^{-*}. \end{aligned}$$

Noting that  $S$  is a contraction - since both  $Z$  and  $A$  are contractions - it follows that  $\mathcal{Z}_n$  is a contraction.

Next observe that, for given  $\mathbf{f} \in H^2(k) \otimes L^2(\mu) \otimes \ell^2$ ,  $z \in \mathbb{A}$  and  $F \in L^2(\mu) \otimes \ell^2$ ,

$$\begin{aligned} \langle \mathcal{Z}_n \mathbf{f}, k_z \otimes F \rangle &= (1 - t_n) \sum_j \langle \mathbf{f}, (t_n(I \otimes A)^* Z^*)^j k_z \otimes F \rangle \\ &= (1 - t_n) \sum_j \langle \mathbf{f}, k_z \otimes (t_n A^* \rho(E(z))^*)^j F \rangle \\ &= (1 - t_n) \langle \mathbf{f}, k_z \otimes (I - t_n A^* \rho(E(z))^*)^{-1} F \rangle \end{aligned}$$

which evidently tends to 0 as  $t_n$  tends to 1, since  $\|\rho(E(z))A\| < 1$ . The statement about WOT convergence now follows.

Let  $\mathbb{W}_n = C(I - t_n \rho(E(z))A)^{-1} \rho(E(z))B$ . Because  $\mathbb{W}_n$  converges pointwise boundedly to  $\mathbb{W}$ ,  $M_{\mathbb{W}_n}$  converges WOT boundedly to  $M_{\mathbb{W}}$ .

Next, for  $m, h \in \mathcal{M}$ ,

$$\begin{aligned} &\langle (I \otimes C)(I - t_n Z(I \otimes A))^{-1} Z(I \otimes B)(1 \otimes m), k_z \otimes h \rangle \\ &= \langle (I - t_n Z(I \otimes A))^{-1} Z 1 \otimes Bm, k_z \otimes C^* h \rangle \\ &= \langle (I - t_n \rho(E(z))A)^{-1} \rho(E(z))Bm, C^* h \rangle \\ &= \langle \mathbb{W}_n(z)m, h \rangle \\ &= \langle (M_{\mathbb{W}_n} 1 \otimes m)(z), h \rangle \\ &= \langle M_{\mathbb{W}_n} 1 \otimes m, k_z \otimes h \rangle. \end{aligned}$$

Hence

$$(I \otimes C)(I - t_n Z(I \otimes A))^{-1} Z(I \otimes B)(1 \otimes m) = M_{\mathbb{W}_n} m.$$

We are now in a position to complete the proof. Using Lemma 7.3,

$$\begin{aligned} &V^* M_{\mathbb{W}_n} (1 \otimes m) \\ &= V^* (I \otimes C)(I - t_n Z(I \otimes A))^{-1} Z(I \otimes B)(1 \otimes m) \\ &= J^* (I \otimes \iota^* \Phi \mathcal{W}^*) [I - Z(I \otimes A)] \times \\ &\quad (I - t_n Z(I \otimes A))^{-1} Z(I \otimes B)(1 \otimes m) \\ &= J^* (I \otimes \iota^* \Phi \mathcal{W}^*) Z(1 \otimes Bm) \\ &\quad + (t_n - 1) V^* (I - t_n Z(I \otimes A))^{-1} Z(I \otimes B)(1 \otimes m). \end{aligned}$$

As  $n$  tends to infinity, the left hand side tends to  $V^* M_{\mathbb{W}}$  (WOT) and the second term on the right hand side tends to 0 (WOT) completing the proof.  $\square$

## 8. PROOF OF (AD) IMPLIES (SC)

Using the ingredients assembled in the previous section, the proof that (ad) implies (sc) follows readily. For  $f \in H^\infty(\mathbb{A})$  and  $m \in \mathcal{M}$ ,

$$\begin{aligned}
 V^*M_W(f \otimes m) &= V^*M_fM_W(1 \otimes m) \quad \text{since } M_f, M_W \text{ commute} \\
 &= T_fV^*M_W(1 \otimes m) \quad \text{since } V^* \text{ intertwines } M_f \text{ and } T_f \\
 &= T_fV^*[(1 \otimes Dm) + M_{\mathbb{W}}(1 \otimes m)] \\
 &= T_fV^*(1 \otimes Dm) + T_fJ^*(I \otimes \iota^*\Phi\mathcal{W}^*)Z1 \otimes Bm \quad \text{from Lemma 7.4} \\
 &= T_fR^*Dm + T_fJ^*(I \otimes \iota^*\Phi\mathcal{W}^*)Z1 \otimes Bm \quad \text{from equation (8)} \\
 &= T_fR^*Dm + T_f\iota^*Y^*\Phi\mathcal{W}^*Bm \quad \text{using equation (25)} \\
 &= T_f[R^*D + Y^*\Phi\mathcal{W}^*B]m \\
 &= T_fXR^*m \quad \text{using the second equation in (22)} \\
 &= XT_fR^*m \\
 &= XV^*(f \otimes m) \quad \text{from Proposition 3.1, equation (8).}
 \end{aligned}$$

## 9. THE CONVERSE

This section is devoted to the proof of the implication (sc) implies (ad) of Theorem 2.9. Accordingly assume hypotheses (i), (ii), and (iii) and also the representation (sc) for  $X$  in Theorem 2.9 throughout this section. Thus there is an  $W$  with a  $\Psi$ -unitary colligation transfer function representation

$$W(z) = D + C(I - \rho(E(z))A)^{-1}\rho(E(z))B$$

such that  $VX^* = M_W^*V$ .

For definiteness, write

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathcal{E} \\ \oplus \\ \mathcal{H} \end{matrix} \rightarrow \begin{matrix} \mathcal{E} \\ \oplus \\ \mathcal{H} \end{matrix}.$$

For technical reasons, let, for  $0 \leq r < 1$ ,

$$W_r(z) = D + C(I - r\rho(E(z))A)^{-1}r\rho(E(z))B.$$

Like before, let

$$H_r(z) = C(I - r\rho(E(z))A)^{-1}.$$

The usual computation reveals,

$$(26) \quad I - W_r(z)W_r(w)^* = H_r(z)(I - r^2\rho(E(z))\rho(E(w))^*)H_r(w)^*.$$

There is a spectral measure  $\mathbb{E}$  associated with the representation  $\rho$ . Thus  $\mathbb{E} : \mathfrak{B}(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{E})$  and, in particular,

$$\rho(E(z))\rho(E(w))^* = \int_{\Psi} E(z)E(w)^*d\mathbb{E}(\psi) = \int_{\Psi} \psi(z)\psi(w)^*d\mathbb{E}(\psi),$$

where  $E(z)(\psi) = \psi(z)$  has been used.

**Lemma 9.1.** *There exists a constant  $\kappa > 0$  so that*

$$H(z)H(w)^* \preceq \kappa k(z, w).$$

(Here the inequality is in the sense of kernels).

*Proof.* Multiplying equation (26) by  $k(z, w)$  gives,  
(27)

$$(I - W_r(z)W_r(w)^*)k(z, w) = H_r(z) \left[ \int k(z, w)(1 - r^2\psi(z)\psi(w)^*)dE(\psi) \right] H_r(w)^*$$

On the other hand, with  $b = \sqrt{q}$ , since  $\psi(b) = 0$  we have

$$k(z, b)(1 - r^2\psi(z)\psi(b)^*) = k(z, b).$$

Thus,

$$(28) \quad k(z, w)(1 - r^2\psi(z)\psi(w)^*) \succeq \frac{k(z, b)k(b, w)}{k(b, b)}.$$

Letting  $G(z) = \frac{k(z, b)}{\sqrt{k(b, b)}}$ , combining equations (28) and (27), using

$$k(z, w) \succeq k(z, w)(I - W_r(z)W_r(w)^*)$$

and  $\mathbb{E}(\mathbb{T}) = I$ , gives,

$$(29) \quad k(z, w) \succeq H_r(z)G(z)G(w)^*H_r(w)^*.$$

The function  $g(z) = \frac{1}{G}$  is analytic in a neighborhood of the annulus and is thus a multiplier of  $H^2(k)$ . In particular, there is an  $\eta$  so that  $k(z, w)(\eta^2 - g(z)g(w)^*) \succeq 0$ . This last inequality is more conveniently written as

$$(30) \quad \eta^2 k(z, w) \succeq k(z, w)g(z)g(w)^*.$$

Putting equations (29) and (30) together yields,

$$\eta^2 k(z, w) \succeq k(z, w)g(z)g(w)^* \succeq H_r(z)H_r(w)^*$$

□

From Lemma 9.1 it follows that  $H_r$  induces a bounded linear operator  $\mathbb{H}_r^* : H^2(k) \otimes \mathcal{H} \rightarrow \mathcal{E}$  of norm at most  $\sqrt{\kappa}$ , determined by

$$\mathbb{H}_r^* k_z \otimes e = H_r(z)^* e.$$

Hence for  $0 < r \leq 1$ , the formula

$$\mathbb{Q}_r(\omega) = \mathbb{H}_r \mathbb{E}(\omega) \mathbb{H}_r^*,$$

for a Borel subset  $\omega$  of  $\mathbb{T}$ , defines a  $\mathcal{B}(\mathcal{H}^2(k) \otimes \mathcal{H})$ -valued measure.

Let  $\mathbb{Q} = \mathbb{Q}_1$ . Define  $\mu_r(\omega) = V^* \mathbb{Q}_r(\omega) V$  and let  $\mu = \mu_1$ . Finally, let  $\Lambda_r(\omega) = k(T, T^*)(\mu_r(\omega))$  and  $\Lambda = \Lambda_1$ .

**Lemma 9.2.** *For fixed  $\omega$ , the operators  $\mathbb{Q}_r(\omega)$  are uniformly bounded by  $\kappa$ , and, for each  $\omega$ , the net  $\mathbb{Q}_r(\omega)$  converges WOT to  $\mathbb{Q}(\omega)$ .*

*Similarly,*

$$\mu(\omega) = V^* \mathbb{Q}(\omega) V$$

*defines a  $\mathcal{B}(\mathcal{M})$ -valued measure on  $\mathbb{T}$  and  $\mu_r(\omega) = V^* \mathbb{Q}_r(\omega) V$  converges WOT boundedly to  $\mu(\omega)$ .*

*Finally,  $\Lambda_r(\mathbb{T})$  is uniformly bounded and  $\Lambda_r(\omega)$  converges WOT to  $\Lambda(\omega)$  for each Borel set  $\omega$ .*



*Proof.* The uniform bound on the  $\mathbb{Q}_r$  follows immediately from the fact that  $\mathbb{H}_r$  is uniformly bounded. Next, as  $r$  tends to 1,

$$\langle \mathbb{Q}_r(\omega)k_z \otimes e, k_w \otimes f \rangle = \langle \mathbb{E}(\omega)H_r(z)^*e, H_r(w)^*f \rangle \rightarrow \langle \mathbb{E}(\omega)H_1(z)^*e, H_1(w)^*f \rangle.$$

Here we have used, for a fixed  $z \in \mathbb{A}$ ,  $H_r(z)$  converges in norm to  $H_1(z)$ . Since  $\mathbb{Q}_r(\omega)$  is uniformly bounded and converges WOT to  $\mathbb{Q}(\omega)$  against a dense set of vectors, it converges WOT to  $\mathbb{Q}(\omega)$ .

The spectral condition on  $T$  and the fact that  $\mu_r(\mathbb{T})$  is uniformly bounded implies  $\Lambda_r(\mathbb{T})$  is also uniformly bounded. Since  $\mu_r(\omega)$  converges WOT to  $\mu(\omega)$  it follows that  $k(T, T^*)(\mu_r(\omega))$  converges WOT to  $k(T, T^*)(\mu(\omega))$ .  $\square$

To complete the proof of Theorem 2.9 it remains to show that  $\mu$  produces an Agler decomposition for the pair  $(T, X)$ .

**Lemma 9.3.** *For  $0 < r < 1$  and each Borel set  $\omega \subset \mathbb{T}$ ,*

$$\Lambda_r(\omega) = V^*M_{H_r}(I \otimes \mathbb{E}(\omega))M_{H_r}^*V.$$

*For any  $\varphi \in H^\infty(\mathbb{A})$  of norm at most one and Borel set  $\omega$ ,*

$$\Lambda(\omega) - T_\varphi \Lambda(\omega) T_\varphi^* \succeq 0.$$

Using the definition of  $\Lambda$ , the conclusion of the second part of the lemma is

$$k(T, T^*)(\mu(\omega) - T_\varphi \mu(\omega) T_\varphi^*) \succeq 0.$$

From the definition of  $\mu$  and the lifting property  $VT_\psi^* = M_\psi^*V$ ,

$$\mu(\omega) - T_\varphi \mu(\omega) T_\varphi^* = V^*(\mathbb{Q}(\omega) - M_\varphi \mathbb{Q}(\omega) M_\varphi^*)V.$$

*Proof.* Let

$$k_n(z, w) = \sum_{-n}^n \zeta_j(z) \zeta_j(w)^*$$

so that for an operator  $G$ ,

$$k_n(M, M^*)(G) = \sum_{-n}^n M_j G M_j^*,$$

where  $M_j = M_{\zeta_j}$ . To prove that  $k_n(M, M^*)(\mathbb{Q}_r(\omega))$  converges WOT to  $M_{H_r}(I \otimes \mathbb{E}(\omega))M_{H_r}^*$ , observe, for a Borel set  $\omega$ , that

$$\begin{aligned} \langle k_n(M, M^*)(\mathbb{Q}_r(\omega))k_w \otimes e, k_z \otimes f \rangle &= k_n(z, w) \langle \mathbb{E}(\omega)H_r(w)^*e, H_r^*(z)f \rangle \\ &\preceq k(z, w) \langle \mathbb{E}(\omega)H_r(w)^*e, H_r^*(z)f \rangle \\ (31) \quad &= \langle M_{H_r}(I \otimes \mathbb{E}(\omega))M_{H_r}^*k_w \otimes e, k_z \otimes f \rangle \\ &\preceq \langle M_{H_r}M_{H_r}^*k_w \otimes e, k_z \otimes f \rangle, \end{aligned}$$

where the inequalities are in the sense of (positive semidefinite) kernels.

Since also  $k_n(M, M^*)(\mathbb{Q}_r(\omega))$  is a bounded increasing sequence of positive operators equation (31) implies that  $k_n(M, M^*)(\mathbb{Q}_r(\omega))$  converges WOT to  $M_{H_r}(I \otimes \mathbb{E}(\omega))M_{H_r}^*$ . Hence,  $V^*k_n(M, M^*)(\mathbb{Q}_r(\omega))V = k_n(T, T^*)(\mu_r(\omega))$  converges to  $V^*M_{H_r}(I \otimes \mathbb{E}(\omega))M_{H_r}^*V$ , proving the first part of the Lemma.

Similarly,  $k_n(M, M^*)(M_\varphi \mathbf{Q}_r(\omega) M_\varphi^*) = M_\varphi k_n(M, M^*)(\mathbf{Q}_r(\omega)) M_\varphi^*$ , converges WOT to  $M_\varphi M_{H_r}(I \otimes \mathbb{E}(\omega)) M_{H_r}^* M_\varphi^*$ . Thus, letting  $n$  tend to infinity and using the definition of  $\mathbf{Q}_r$ ,

$$\begin{aligned} & \langle k_n(M, M^*)(\mathbf{Q}_r(\omega) - M_\varphi \mathbf{Q}_r(\omega) M_\varphi^*) k_w \otimes e, k_z \otimes f \rangle \\ & \rightarrow (1 - \varphi(z) \varphi(w)^*) k(z, w) \langle H_r(z) \mathbb{E}(\omega) H_r(w)^* e, f \rangle \end{aligned}$$

The kernel on the right hand side is positive semi-definite because it is the pointwise product of positive semi-definite kernels, and thus

$$\lim_{\text{WOT}} k_n(M, M^*)[\mathbf{Q}_r(\omega) - M_\varphi \mathbf{Q}_r(\omega) M_\varphi^*] \succeq 0.$$

Thus,

$$\begin{aligned} 0 & \preceq V^* \lim_{\text{WOT}} k_n(M, M^*)(\mathbf{Q}_r(\omega) - M_\varphi \mathbf{Q}_r(\omega) M_\varphi^*) V \\ & = k(T, T^*)[V^* \mathbf{Q}_r(\omega) V - T_\varphi V^* \mathbf{Q}_r(\omega) V T_\varphi^*]. \end{aligned}$$

Finally, letting  $r$  tend to 1 on the right hand side above and applying Lemmas 9.2 and 2.5 gives

$$0 \preceq k(T, T^*)[\mu(\omega) - T_\varphi \mu(\omega) T_\varphi^*].$$

□

It remains to verify the condition of equation (5). The argument is an elaboration on the proof of the preceding lemma, making use of the approximations  $H_r$  and the related operators  $\mathbb{H}_r$  and  $M_{H_r}$ .

We break the proof into several steps as outlined in the Lemma below.

**Lemma 9.4.** *With notations as above:*

(i) *For  $0 < r < 1$  and each Borel set  $\omega \subset \mathbb{T}$ ,*

$$\Lambda_r(\omega) = V^* M_{H_r}(I \otimes \mathbb{E}(\omega)) M_{H_r}^* V$$

*and converges WOT to  $\Lambda(\omega)$ .*

(ii) *There is a constant  $C_*$  such that  $\|\Lambda_r(\mathbb{T})\| \leq C_*^2$  for all  $0 < r < 1$ . In particular,  $\|M_{H_r}^* V\| \leq C_*$  independent of  $r$ .*

(iii) *There is a bounded operator  $\Gamma$  on  $H^2(k) \otimes \mathcal{E}$  determined by*

$$\langle \Gamma k_w \otimes f, k_z \otimes g \rangle = k(z, w) \int \psi(z) \psi(w)^* d \langle \mathbb{E}(\psi) f, g \rangle.$$

(iv) *If  $(\omega_j)$  is a Borel partition of  $\mathbb{T}$  of diameter at most  $\epsilon > 0$ , then for any choice of points  $s_j \in \omega_j$ ,*

$$\epsilon > \|\Gamma - \sum M_{s_j} M_{s_j}^* \otimes \mathbb{E}(\omega_j)\|.$$

(v) *The identity*

$$k(T, T^*) \left( \int T_\psi d\mu(\psi) T_\psi^* \right) = \int T_\psi d\Lambda(\psi) T_\psi^*$$

*holds.*

Note, in item (iv) the identification of  $\mathbb{T}$  with  $\Psi$  is in force so that  $|s - t| < \epsilon$  means  $\|M_s - M_t\| = \|s - t\|_\infty < \epsilon$ .

*Proof of Lemma 9.4.* The first part of item (i) is part of Lemma 9.2. The description of  $\Lambda_r$  in terms of  $M_{H_r}$  and  $\mathbb{E}$  is the first part of Lemma 9.3.

To prove item (ii), note that  $\Lambda_r(\mathbb{T})$  is uniformly bounded by Lemma 9.2, so there is a  $C_*$ . The bound on  $M_{H_r}^*$  follows from this bound on  $\Lambda_r(\mathbb{T})$  and the representation of  $\Lambda_r$  in item (i).

Item (iii) is a consequence of the fact that, as kernels,  $k(z, w)\psi(z)\psi(w)^* \preceq k(z, w)$ .

To prove item (iv), first note if  $(\omega_j)_j$  is a partition of  $\mathbb{T}$ , then

$$\sum \Gamma(\omega_j) = \Gamma.$$

Next observe that if  $s, t \in \Psi$  and  $|s - t| < \epsilon$ , then, since also  $\|M_s\|, \|M_t\| = 1$ , we have  $\|M_s M_s^* - M_t M_t^*\| < \epsilon$ .

To finish the proof of item (iv), choose any partition  $(\omega_j)$  of  $\Psi = \mathbb{T}$  of width at most  $\epsilon > 0$ . Thus, if  $s, t \in \omega_j$ , then  $\|M_s - M_t\| < \epsilon$ ; i.e., the sup norm of the difference of the functions  $s, t : \mathbb{A} \rightarrow \mathbb{D}$  is less than  $\epsilon$ . Thus, if  $s_j, t_j \in \omega_j$ , then

$$\left\| \sum M_{s_j} M_{s_j}^* \otimes \mathbb{E}(\omega_j) - \sum M_{t_j} M_{t_j}^* \otimes \mathbb{E}(\omega_j) \right\| \leq 2\epsilon.$$

Consequently, choosing a sequence of partitions such that the width of the partitions tends to zero, the corresponding Riemann sums form a norm Cauchy sequence and thus converge to some operator. At the same time, this sequence converges WOT to  $\Gamma$ , since

$$\left\langle \sum M_{s_j} M_{s_j}^* \otimes \mathbb{E}(\omega_j) k_w \otimes f, k_z \otimes g \right\rangle = \sum_j s_j(z) s_j(w)^* k(z, w) \langle d\mathbb{E}(\omega_j) f, g \rangle.$$

Thus the sequence of Riemann sums converges in norm to  $\Gamma$ . Comparing any Riemann sum whose partition has width at most  $\epsilon > 0$  with an appropriate term of the sequence just constructed completes the proof of (iv).

From Lemma 4.7 and Remark 4.8, the Riemann sums  $\Delta(P, S, \mu)$  and  $\Delta(P, S, \Lambda)$  converge WOT to  $\int T_\psi d\mu(\psi) T_\psi^*$  and  $\int T_\psi d\Lambda(\psi) T_\psi^*$  respectively. Hence the net  $k(T, T^*)(\Delta(P, S, \mu))$  converges to the RHS of item (v). On the other hand, we have  $k(T, T^*)(\Delta(P, S, \mu)) = \Delta(P, S, \Lambda)$ . Hence  $k(T, T^*)(\Delta(P, S, \mu))$  converges WOT to both the right and left hand side of (v) and the result follows.  $\square$

Using Lemma 9.4, the proof that (sc) implies equation (5), and hence the converse of Theorem 2.9, proceeds as follows. From Lemma 9.4 and the representation

$$(I - W_r(z)W_r(w)^*)k(z, w) = H_r(z) \left[ \int_{\Psi} (1 - r^2 \psi(z)\psi(w)^*) k(z, w) d\mathbb{E}(\psi) \right] H_r(w)^*$$

it follows that

$$V^*(I - M_{W_r} M_{W_r}^*)V = V^* M_{H_r} [I - r^2 \Gamma] M_{H_r}^* V.$$

The left hand side converges WOT to  $I - XX^*$  (because  $M_W^* V = V M_W^*$ ) and thus so does

$$V^* M_{H_r} (I - \Gamma) M_{H_r}^* V.$$

Since, by item (i) of Lemma 9.4 with  $\omega = \mathbb{T}$ ,  $V^* M_{H_r} M_{H_r}^* V$  converges WOT to  $\Lambda(\mathbb{T})$ , it follows that

$$(32) \quad V^* M_{H_r} \Gamma M_{H_r}^* V \rightarrow \Lambda(\mathbb{T}) - I + XX^*$$

WOT.

Fix a vector  $m \in \mathcal{M}$ . Given  $\epsilon > 0$ , choose, using (iv) of Lemma 9.4 and Lemma 4.7 respectively, a tagged partition  $(P, S)$  such that both,

$$(33) \quad \begin{aligned} \frac{1}{\|m\| + 1} \epsilon &> \left\| \sum M_{s_j} M_{s_j}^* \otimes \mathbb{E}(\omega_j) - \Gamma \right\| \\ \epsilon &> \left| \left\langle \int T_\psi d\Lambda(\psi) T_\psi^* m, m \right\rangle - \sum \langle T_{s_j} \Lambda(\omega_j) T_{s_j}^* m, m \rangle \right|. \end{aligned}$$

Using item (i) of Lemma 9.4 and equation (32) respectively, choose  $0 < r_0 < 1$  (depending upon  $(P, S)$ ) such that for  $r_0 \leq r < 1$ ,

$$(34) \quad \begin{aligned} \epsilon &> \left| \sum \langle T_{s_j} \Lambda(\omega_j) T_{s_j}^* m, m \rangle - \sum \langle T_{s_j} \Lambda_r(\omega_j) T_{s_j}^* m, m \rangle \right| \\ \epsilon &> \left| \langle V^* M_{H_r} \Gamma M_{H_r}^* V m, m \rangle - \langle (\Lambda(\mathbb{T}) - I + X X^*) m, m \rangle \right|. \end{aligned}$$

Note that combining the first inequality in equation (33) with item (ii) of Lemma 9.4 gives,

$$(35) \quad \begin{aligned} & \left| \langle V^* M_{H_r} \sum M_{s_j} M_{s_j}^* \otimes \mathbb{E}(\omega_j) M_{H_r}^* V m, m \rangle - \langle V^* M_{H_r} \Gamma M_{H_r}^* V m, m \rangle \right| \\ &= \left| \langle V^* M_{H_r} \left[ \sum M_{s_j} M_{s_j}^* \otimes \mathbb{E}(\omega_j) - \Gamma \right] M_{H_r}^* V m, m \rangle \right| < C_*^2 \epsilon. \end{aligned}$$

Similarly, observe that

$$(36) \quad \begin{aligned} \sum \langle T_{s_j} \Lambda_r(\omega_j) T_{s_j}^* m, m \rangle &= \sum \langle T_{s_j} V^* M_{H_r} (I \otimes \mathbb{E}(\omega_j)) M_{H_r}^* V T_{s_j}^* m, m \rangle \\ &= \sum \langle V^* M_{H_r} M_{s_j} (I \otimes \mathbb{E}(\omega_j)) M_{s_j}^* M_{H_r}^* V m, m \rangle \\ &= \langle V^* M_{H_r} \left[ \sum M_{s_j} (I \otimes \mathbb{E}(\omega_j)) M_{s_j}^* \right] M_{H_r}^* V m, m \rangle. \end{aligned}$$

Putting it all together, it follows from (34), (35), and (36) that

$$\begin{aligned} & \left| \left\langle \int T_\psi d\Lambda(\psi) T_\psi^* m, m \right\rangle - \langle (\Lambda(\mathbb{T}) - I + X X^*) m, m \rangle \right| \\ & \leq \left| \left\langle \int T_\psi d\Lambda(\psi) T_\psi^* m, m \right\rangle - \sum \langle T_{s_j} \Lambda(\omega_j) T_{s_j}^* m, m \rangle \right| \\ & \quad + \left| \sum \langle T_{s_j} \Lambda(\omega_j) T_{s_j}^* m, m \rangle - \sum \langle T_{s_j} \Lambda_r(\omega_j) T_{s_j}^* m, m \rangle \right| \\ & \quad + \left| \langle V^* M_{H_r} \left[ \sum M_{s_j} (I \otimes \mathbb{E}(\omega_j)) M_{s_j}^* - \Gamma \right] M_{H_r}^* V m, m \rangle \right| \\ & \quad + \left| \langle V^* M_{H_r} \Gamma M_{H_r}^* V m, m \rangle - \langle (\Lambda(\mathbb{T}) - I + X X^*) m, m \rangle \right| \\ & < \epsilon + \epsilon + C_* \epsilon + \epsilon. \end{aligned}$$

Thus,

$$I - X X^* = \Lambda(\mathbb{T}) - \int T_\psi d\Lambda(\psi) T_\psi^*.$$

An application of item (v) of Lemma 9.4 completes the proof.

## 10. DETAILS ON THE KERNEL

This section gives the details on the basic facts about our kernel  $k$ . It requires a digression into theta functions much of which is borrowed from [28].

Begin by recalling the theta function

$$\vartheta_1(x) = \vartheta_1(x, q) = 2q^{\frac{1}{4}} \sin(x) \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n} e^{2ix})(1 - q^{2n} e^{-2ix}),$$

and the Jordan-Kronecker function

$$f(\alpha, p) = \sum_{n=-\infty}^{\infty} \frac{\alpha^n}{1 - pq^{2n}}.$$

It is well known that these functions are related by

$$f(\alpha, p) = C \frac{\vartheta_1(x+y)}{\vartheta_1(x)\vartheta_1(y)},$$

where  $x$  and  $y$  are chosen so that  $\alpha = e^{2ix}$  and  $p = e^{2iy}$  and  $C$  is a constant (independent of  $x, y$ ).

Replacing  $p$  with  $-t$  and thus  $y$  with  $y + \frac{\pi}{2}$  and letting  $\alpha = zw^*$  gives,

$$k(z, w; t) = C \frac{\vartheta_1(x+y+\frac{\pi}{2})}{\vartheta_1(x)\vartheta_1(y+\frac{\pi}{2})}$$

From its product expansion, it is evident that the zeros of  $\vartheta_1$  are  $q^{2m} = e^{2ix}$  for integers  $m$  and thus  $k(z, w; t) = 0$  if and only if  $tz w^* = -q^{2m}$  for some integer  $m$ . Thus, unless  $t = q^{2\ell}$  for some  $\ell$ , there exists points  $z, w \in \mathbb{A}$  such that  $k(z, w; t) = 0$ .

We are interested in the case  $t = 1$  ( $p = -1$  and  $y = 0$  above) which gives,

$$k(z, w; 1) = k(z, w) = C \frac{\vartheta_1(x+\frac{\pi}{2})}{\vartheta_1(x)\vartheta_1(\frac{\pi}{2})}.$$

In particular,  $k(z, w)$  vanishes if and only if  $zw^* = -q^{2m}$ . In particular,  $k(z, w)$  does not vanish for both  $z$  and  $w$  in the annulus, and further for each fixed  $w \in \mathbb{A}$ , as a function of  $z$ , the kernel  $k(z, w)$  extends beyond the annulus to a meromorphic function.

If  $zw^* = e^{2ix}$ , then  $-zw^* = e^{2i(x+\frac{\pi}{2})}$  and therefore,

$$\begin{aligned} k(z, -w) &= C \frac{\vartheta_1(x+\pi)}{\vartheta_1(x+\frac{\pi}{2})\vartheta_1(\frac{\pi}{2})} \\ &= -C \frac{\vartheta_1(x)}{\vartheta_1(x+\frac{\pi}{2})\vartheta_1(\frac{\pi}{2})} \\ &= C' \frac{1}{k(z, w)}, \end{aligned}$$

where  $C' = \theta_1(\frac{\pi}{2})^{-2}$ . It is evident that  $C' > 0$ .

## 11. DETAILS ON THE TEST FUNCTIONS

Generally the minimal inner functions on a multiply connected domain can be constructed using the Green's functions or as a product of quotients of theta functions. In the case of the annulus the first construction is relatively simple to describe, given unique solutions to the Dirichlet problem.

The first step is to construct, given a point  $a \in \mathbb{A}$ , an analytic function with modulus one on the outer boundary  $B_0$  and constant modulus on the inner boundary  $B_1$  with just one zero, at  $a$ , in  $\mathbb{A}$ . There is a harmonic function  $w$  whose boundary values (on the boundary of  $\mathbb{A}$ ) agree with the boundary values of  $\log|z - a|$ . There is a constant  $\beta$  and an analytic function  $f$  on  $\mathbb{A}$  so that

$$w = \Re(f + \beta \log(|z|)).$$

Here  $\Re$  denotes the real part. Note that  $\beta$  can be computed because a harmonic function  $u = w - \beta \log(|z|)$  is the real part of an analytic function on  $\mathbb{A}$  if and only

if the integral of  $u$  around the outer boundary agrees with the integral of  $u$  around the inner boundary of  $\mathbb{A}$ ; i.e.,

$$+2\pi\beta\log(q) = \int \log|\exp(it) - a|dt - \int \log|q\exp(it) - a|dt.$$

Indeed, a simple computation shows  $\beta = \frac{\log(|a|)}{\log(q)}$ . In particular, given two points  $a, b \in \mathbb{A}$ , there is a function unimodular on the boundary of  $\mathbb{A}$  with zeros precisely  $a$  and  $b$  (with multiplicity if needed) if and only if  $\log(|ab|) = q$ .

## REFERENCES

- [1] Abrahamse, M.B. *The Pick interpolation theorem for finitely connected domains*. Michigan Math. J. **26** (1979), no. 2, 195–203.
- [2] Abrahamse, M.B.; Douglas, R.G. *A class of subnormal operators related to multiply-connected domains* Advances in Math. **19** (1976), no. 1, 106–148.
- [3] Ambrozie, Calin; Eschmeier, Jörg. *A commutant lifting theorem on analytic polyhedra*. Topological algebras, their applications, and related topics, 83108, Banach Center Publ., 67, Polish Acad. Sci., Warsaw, 2005.
- [4] Ambrozie, C.-G.; Englis, M.; Müller, V. *Operator tuples and analytic models over general domains in  $\mathbb{C}^n$* , J. Operator Theory **47** (2002), no. 2, 287–302.
- [5] Agler, Jim. *On the representation of certain holomorphic functions defined on a polydisc*. Topics in operator theory: Ernst D. Hellinger memorial volume, 47–66, Oper. Theory Adv. Appl., **48**, Birkhäuser, Basel, 1990.
- [6] Agler, Jim. *The Arveson extension theorem and coanalytic models*. Integral Equations Operator Theory **5** (1982), no. 1, 608–631.
- [7] Agler, Jim; McCarthy, John E. *Nevanlinna-Pick interpolation on the bidisk*. J. Reine Angew. Math. **506** (1999), 191–204.
- [8] Agler, Jim; McCarthy, John E. *Pick interpolation and Hilbert function spaces*. Graduate Studies in Mathematics, **44**. American Mathematical Society, Providence, RI, 2002. xx+308 pp. ISBN: 0-8218-2898-3.
- [9] Agler, Jim; McCarthy, John E. *Complete Nevanlinna-Pick kernels*. J. Funct. Anal. **175** (2000), no. 1, 111–124.
- [10] Ambrozie, C.-G. *Remarks on the operator-valued interpolation for multivariable bounded analytic functions*. Indiana Univ. Math. J. **53** (2004), no. 6, 1551–1576.
- [11] Archer, Robert. *Unitary dilations of commuting contractions*. PhD thesis, University of Newcastle, 2004.
- [12] Ball, Joseph A.; Bolotnikov, V. Vladimir *Realization and interpolation for Schur-Agler-class functions on domains with matrix polynomial defining function in  $\mathbb{C}^n$* . J. Funct. Anal. **213** (2004), no. 1, 45–87.
- [13] Beatrous, Frank; Burbea, Jacob. *Reproducing kernels and interpolation of holomorphic functions*. Complex analysis, functional analysis and approximation theory (Campinas, 1984), 25–46, North-Holland Math. Stud., 125, North-Holland, Amsterdam, 1986.
- [14] Ball, J. A.; Li, W. S.; Timotin, D.; Trent, T. T. *A commutant lifting theorem on the polydisc*. Indiana Univ. Math. J. **48** (1999), no. 2, 653–675.
- [15] Ball, Joseph A.; Trent, Tavan T. *Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna-Pick interpolation in several variables*. J. Funct. Anal. **157** (1998), no. 1, 1–61.
- [16] Ball, Joseph A.; Trent, Tavan T.; Vinnikov, Victor. *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*. Operator theory and analysis (Amsterdam, 1997), 89–138, Oper. Theory Adv. Appl., **122**, Birkhauser, Basel, 2001.
- [17] de Branges, Louis; Rovnyak, James. *Canonical models in quantum scattering theory*. 1966 Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965) pp. 295–392 Wiley, New York
- [18] Dritschel, Michael A.; McCullough, Scott. *The failure of rational dilation on a triply connected domain*. J. Amer. Math. Soc. **18** (2005), no. 4, 873918 (electronic).

- [19] Dritschel, Michael A.; McCullough, Scott A., *Test functions, kernels, realizations and interpolation*. Operator theory, structured matrices, and dilations, 153–179, Theta Ser. Adv. Math., **7**, Theta, Bucharest, 2007.
- [20] Dritschel, Michael A.; Marcantognini, Stefania; McCullough, Scott. *Interpolation in Semigroupoid Algebras*. Journal für die Reine und Angewandte Mathematik 2006.
- [21] Douglas, R. G.; Muhly, P. S.; Pearcy, Carl. *Lifting commuting operators*. Michigan Math. J. **15** 1968 385–395.
- [22] Eschmeier, Jörg. *Tensor products and elementary operators*. J. Reine Angew. Math. 390 (1988), 4766.
- [23] Foias, Ciprian; Frazho, Arthur E. *The commutant lifting approach to interpolation problems*. Operator Theory: Advances and Applications, **44**, Birkhauser Verlag, Basel, 1990.
- [24] Fritz Gesztesy, Rudi Weikard, Maxim Zinchenko. *On a class of Model Hilbert Spaces*. arXiv:1111.0645v1.
- [25] Jury, M.T. *An improved Julia-Caratheodory theorem for Schur-Agler mappings of the unit ball*. arXiv:0707.3423v1.
- [26] McCullough, Scott, *Isometric representations of some quotients of  $H^\infty$  of an annulus*. Integral Equations Operator Theory 39 (2001), no. 3, 335–362.
- [27] McCullough, Scott *The trisecant identity and operator theory*. Integral Equations Operator Theory 25 (1996), no. 1, 104–127.
- [28] McCullough, Scott; and Shen, Li Chien, *On the Szegő's kernel of an annulus*. Proc. Amer. Math. Soc. 121 (1994), no. 4, 1111–1121.
- [29] McCullough, Scott; and Sultanic, Saida *Ersatz commutant lifting with test functions*. Complex Anal. Oper. Theory 1 (2007), no. 4, 581–620.
- [30] Sz.-Nagy, Bela. *Sur les contractions de l'espace de Hilbert*. Acta Sci. Math. Szeged **15**, (1953). 87–92
- [31] Sarason, Donald. *Generalized interpolation in  $H^\infty$* . Trans. Amer. Math. Soc. **127** 1967 179–203.

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